5th International Conference on Applied Informatics
Eger, Hungary, January 28–Ferbuary 3, 2001.

Analysing the Structure of the Set of Neighbourhood Sequences

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Abstract

In this paper we introduce some well-known structural concepts on the set of generalized neighbourhood sequences. Our main attempt is to perform the structural analyses by introducing a velocity function and a metric on this set. The natural ordering compares the sequences according to the number of steps required to reach one point from another. Intuitively, we can use the term "faster" if a sequence is larger than another one with respect to the ordering relation. This consideration led us to define a velocity concept first, then define a metric by using this new concept of velocity.

Categories and Subject Descriptors: I.4 [Image Processing and Computer Vision]; I.5.5 [Pattern Recognition]: Special Architectures;

Key Words and Phrases: Digital Geometry, Neighbourhood Sequences, Distance, Metric

1 Introduction

The structural analysis of the set of neighbourhood sequences was begun by introducing two types of ordering relations [1]. The first one compares two neighbourhood sequences in a natural way, but it gives poor structural results with respect to lattice theory [3]. The second ordering relation has better results according to structural analyses, but has less practical and intuitive meaning.

Since practical applicability is evident in digital image processing, we involve the natural ordering into our further investigations, in spite of its unkind properties with respect to lattice theory. Our main attempt is to go on with our structural analyses by introducing a metric on the set of the generalized neighbourhood sequences. The natural ordering compares the sequences according to the number of steps required to reach one point from another. Intuitively, we can use the term "faster" if a sequence is larger than another one under the ordering relation. This consideration led us to define a concept of velocity, which shows the speed of a neighbourhood sequence. Using this new concept of velocity, we introduce a metric on the set of neighbourhood sequences.

After introducing the metric, the usual investigations are performed, like checking the completeness of the metric space, searching for the dense subsets of the metric space, investigating the topological properties, and so on.

2 Basic concepts and definitions

In this section, we recall [1,2] some definitions about neighbourhood sequences that we need in our forecoming analysis.

Definition 2.1 *The infinite sequence $A = (a_i)_{i=1}^{\infty}$, where $a_i \in \{1, 2, \ldots, n\}$ for all $i \in N$, is called a generalized nD-neighbourhood sequence. Let S_n be the set of the generalized nD-neighbourhood sequences.*

Remark 2.1 *We recall [1,2] the notation*

$$a_i(j) = \begin{cases} j & \text{if } a_i > j, \\ a_i & \text{otherwise.} \end{cases}$$

$$f_i(j) = \begin{cases} \sum_{k=1}^{i} a_k(j) & \text{if } i \geq 1, \\ 0 & \text{if } i = 0. \end{cases}$$

A natural ordering relation on the set of the neighbourhood sequences is defined by comparing these $f_i(j)$ values for two neighbourhood sequences [3].

Definition 2.2 *Let $A, B \in S_n$. The $f_i(j)$ subsum values of the neighbourhood sequences A and B are denoted by $f_i^A(j)$ and $f_i^B(j)$, respectively. We define the relation $\sqsupseteq^*$ in the following way:*

$$A \sqsupseteq^* B \quad \Leftrightarrow \quad f_i^A(j) \geq f_i^B(j),$$

for all $i \in N$ and $j \in \{1, \ldots, n\}$. We say that A is faster than B, if $A \sqsupseteq^* B$.

3 Preliminaries to introduce metric and velocity

When we define velocity, we assign a real number to every neighbourhood sequence. It means, we should define a real valued function, which can be given in several ways. Using the concept of velocity, we define a metric on the set of neighbourhood sequences, which can be done in many ways, as well. The construction of velocity and metric is a bit like the traditional procedure of introducing a norm on the space first, then deducing a metric from the norm. Our construction is similar to this procedure, since we deduce the

metric from the velocity function. The main difference we have here is that the velocity function does not meet the conditions to be a norm.

Since the velocity function and the distance of two neighbourhood sequences can be defined in several ways, we try to take some natural preconditions under consideration to obtain an intuitively obvious velocity and distance function. In this section we list our preconditions, and explain in details the importance of each of these points.

1. *It is not the particular elements of the sequence that are important, but the average behavior of the whole neighbourhood sequence.*
   This condition expresses that we do not focus on the individual elements of the neighbourhood sequence, but try to consider the sequence as a whole. According to this consideration, we use average values in the definitions, instead of the particular elements.

2. *It is recommended to weigh the members of the sequence with a suitable weight function.*
   We have two reasons to establish this precondition, which are in close connection. First, we probably have to emphasize the initial elements of the sequence more, than the elements, which occur later in the sequence. It is because in practical applications, presumably the first elements have greater importance, since this is the beginning of the moving, and we do not take much care about the behavior of the neighbourhood sequence after a large number of steps. The second reason, why we weight the elements, comes from theoretical necessity. Namely, to measure the velocity of a neighbourhood sequence, we have to consider the sum of its elements, or a similar measure. Since neighbourhood sequences have positive integer elements, we have to guarantee the convergence of a such a serie. Weighting the elements is the most natural way to obtain convergence limit for these sequences. As we shall see it later, we need to use weight functions that tend to 0. Using a weight function like that, also have the meaning that we attach more importance to the starting elements of a neighbourhood sequence.

3. *Velocity should define with respect to the "faster" ordering.*
   This is a very natural condition, which requires that velocity should preserve the ordering relation, introduced above. If a neighbourhood sequence is "faster" than another one, its velocity should be larger, as well.

4. *Velocity should be sensitive to dimension.*
   The concept of velocity can be relative according to the dimension, in which we consider the motion. Though a sequence can be faster in higher dimension, than another one, it may happen that they have the same velocity in a lower dimensional subspace. For example, in 3D the sequences (3,3,3,3,...) and (2,2,2,2,...) have the same velocity on the planes, defined by the coordinate axes, or the sequences (1,3,1,3,...) and (2,2,2,2,...) behave differently in different subspaces.

5. *The metric on the set of the neighbourhood sequences should be defined by the concept of velocity.*
   Similarly to the introduction of deducing a metric from a norm, we deduce the metric from the velocity function by taking the difference between velocity values of the two neighbourhood sequences.

4 Assigning velocity to neighbourhood sequences

Considering precondition 4, discussed in Section 3, we define the concept of velocity in two steps. First, we define an nD velocity vector for every neighbourhood sequence, which elements reflect the velocity of the given neighbourhood sequence in the lower dimensional subspaces, in dimensions from 1 to n. Then, to reach our main aim, we make the average value of these n vector elements to have one descriptive velocity value for every neighbourhood sequence.

Definition 4.1 *Let A be an nD-neighbourhood sequence. The weighted velocity of the sequence A in dimension j is defined as*

$$v_j^A = \sum_{k=1}^{\infty} \frac{f_k^A(j)}{k} \delta(k),$$

where $j \in \{1, \ldots, n\}$, and $\sum_{k=1}^{\infty} \delta(k) < \infty$, with $\delta(k) > 0$, for all $k \in N$.

Remark 4.1 *The definition constructed above, has strong relationship with the common velocity concept used in physics. As a natural procedure, velocity is calculated as the ratio of the distance taken and the time. In our definition we use this consideration, since $f_k^A(j)$ is the distance, the neighbourhood sequence A takes after k steps, and the number of steps k is actually the time, the neighbourhood sequence used to take this distance.*

Definition 4.2 *To obtain one descriptive velocity value, we can consider the average value of the weighted velocity values in different dimensions:*

$$v^A = \sum_{j=1}^{n} \frac{v_j^A}{n}.$$

Remark 4.2 *From the definition of v_j^A we can see, how preconditions 1, 2 and 4 are met. Since v^A is deduced from the v_j^A values, v^A also meets these preconditions.*

1. *Since we use the subsum values $f_k^A(j)$ in the definition, we consider the whole sequence instead of the particular elements.*
2. *To weigh the elements of the sequence we use the weight function $\delta(k)$, which provides the convergence of the serie. Moreover, as a necessary condition this weight function has to tend to 0, so the elements that occur later in the sequence have less weight.*
3. *The result is presented in Theorem 4.1.*
4. *The nD velocity vector, defined above, is trivially sensitive to dimension, since we calculate the velocity of a neighbourhood sequence in every lower dimensional subspaces.*

Precondition 3 contains the natural requirement that the sequence A should have larger velocity value than B, if A is "faster" than B ($A \sqsupseteq^* B$). The following theorem shows, that the above definitions of velocity meet this requirement, as well.

Theorem 4.1 *Let A, B be two n-dimensional neighbourhood sequences. Then*

$$A \sqsupseteq^* B \Rightarrow v^A \geq v^B, \quad \text{and} \quad A \sqsupseteq^* B \Rightarrow v_j^A \geq v_j^B, \ \forall j \in \{1, \ \dots \ , n\}.$$

Proof. From the definition of the ordering relation, if $A \sqsupseteq^* B$, then $f_i^A(j) \geq f_i^B(j)$ for all $j \in \{1, \ \dots \ , n\}$, $i \in N$. Thus for every $j \in \{1, \ \dots \ , n\}$ the members of the serie v_j^A are not smaller, than the corresponding members of the serie v_j^B. Since v^A and v^B is the average of these values, respectively, the proof is complete.

Remark 4.3 *The opposite statement of Theorem 4.1 does not hold, since the ordering relation $\sqsupseteq^*$ is only a partial ordering, so not every two neighbourhood sequences can be compared.*

5 Metric space of neighbourhood sequences

According to precondition 5, we introduce a metric on the set of neighbourhood sequences by using the concept of velocity. The way, we introduce this metric, is similar to the common procedure, when a metric is obtained from a norm.

Definition 5.1 *Let A and B be two nD-neighbourhood sequences. The distance of these sequences is defined by the following formula:*

$$\varrho(A, B) = \frac{\sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{|f_k^A(j) - f_k^B(j)|}{k} \ \delta(k)}{n},$$

where $\sum_{k=1}^{\infty} \delta(k) < \infty$, with $\delta(k) > 0$, for all $k \in N$.

Remark 5.1 *The above defined distance function is a metric on the set of the neighbourhood sequences. This fact is an immediate consequence of the properties of the absolute value operation.*

5.1 Topological properties of the metric space

In this subsection we investigate the topological and structural properties of the metric space of the neighbourhood sequences. We check the completeness, searching for the dense subsets of the metric space, and making some more usual analyses.

Theorem 5.1 *(S_n, ϱ) is a complete metric space.*

Proof. We prove the theorem by showing that every Cauchy sequence has a convergence limit. Our proof is constructive, so we show how to obtain this limit sequence. Let $(A_i)_{i=1}^{\infty}$ be a sequence with $\forall i A_i \in S_n$. Suppose that the sequence $(A_i)_{i=1}^{\infty}$ is a Cauchy sequence in (S_n, ϱ). Let $\varepsilon > 0$. Since $(A_i)_{i=1}^{\infty}$ is a Cauchy sequence there exists $k \in N$ such that $\varrho(A_m, A_n) < \varepsilon$ if $m, n > k$. The sequences A_m with $m \geq k$ have the property that their first finitely many elements are equal. The number of the same elements depends on the volume of ε. Let us construct the convergence limit sequence form these same elements. By letting $\varepsilon \to 0$ we can obtain the limit sequence. With this construction, for a given $\varepsilon > 0$ we choose the same $k \in N$ as in the case of the Cauchy sequence to proof the convergence.

Theorem 5.2 *Every bounded monotone sequence* $(A_i)_{i=1}^{\infty}$ *with* $\forall i A_i \in S_n$ *has a convergence limit.*

Proof. The construction of the limit sequence can be performed in the same way as in the proof of Theorem 5.1.

In our next theorem we check, whether the Bolzano property is met in our metric space.

Theorem 5.3 *Every bounded subset of* S_n *with infinite cardinality has an accumulation point.*

Proof. Like in the previous proves, we give an explicit method to construct this accumulation point. Let $S = \{A_i \mid A_i \in S_n, i \in I\}$, where I is a non finite set of indices. For the first element of the accumulation sequence let us choose the element $a_1 \in \{1, \ldots, n\}$ which is the first element of infinitely many $A_i \in S$ sequences. For the kth element ($k \in N$) of the accumulation point let us choose the element $a_k \in \{1, \ldots, n\}$ which is the kth element of infinitely many such $A_i \in S$ sequences, whose first $k-1$ elements are $a_1, a_2, \ldots, a_{k-1}$, respectively.

We go on with searching for the dense subsets of S_n. In the past, only periodic sequences were investigated, so first we check this subset of S_n.

Since the neighbourhood sequences can be modeled by real numbers from the interval $[0, 1]$, we could use real numbers instead of neighbourhood sequences and the interval $[0, 1]$ instead of S_n. A periodic sequence corresponds to a rational number, and a rational number corresponds to an almost periodic neighbourhood sequence (a sequence, which is periodic after ignoring its first finitely many elements).

Theorem 5.4 *The set of periodic neighbourhood sequences is dense in* S_n.

Proof. Let $A \in S_n$ and $\varepsilon > 0$. Let us construct a sequence B in the following way. Let the first k elements of B equal to the first k elements of A, respectively, and let the remaining elements of B arbitrary, such that $\varrho(A, B) < \varepsilon$. Because of the definition of ϱ such a k always exists. Since the elements of B from the index $k+1$ were chosen arbitrarily, thus we can choose these elements, so the sequence B will be periodic, with period k. So we found a periodic sequence B for which $\varrho(A, B) < \varepsilon$, which completes our proof.

Remark 5.2 *Since the set of periodic neighbourhood sequences is a subset of the set of almost periodic neighbourhood sequences, the latter set is also dense in* S_n.

6 Conclusion

Neighbourhood sequences can be effectively used in approximating continuos Euclidian metrics by digital metrics, so they have become important in digital geometry. In this paper we present some new concepts on the set of neighbourhood sequences. By introducing velocity and metric we can compare neighbourhood sequences more precisely, than using only an ordering relation for this comparison. The introduced concepts connect to the structure of neighbourhood sequences in a natural way, and give us new tools to analyse the set, the neighbourhood sequences form.

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