ON POWER VALUES OF POLYNOMIALS

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Abstract. In this paper we give a new, generalized version of a result of Brindza, Evertse and Győry, concerning superelliptic equations.

Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n and b be a nonzero integer. For effective upper bounds obtained by Baker's method for the exponent z in the equation

$$f(x) = by^z, \quad x, y, z \in \mathbb{Z} \text{ with } |y| > 1, \ z > 1 \tag{1}$$

we refer to [T], [ST], [Tu1], [Tu2], [ShT], [B1], [BEGy], [Bu].

For a polynomial P let $M(P)$ denote the Mahler height of it (cf. [M]). The purpose of this paper, which is related to a recent observation of Brindza on the number of solutions of generalized Ramanujan - Nagell equations [B3], is to derive a bound for z which is polynomial in $M(f)$. For brevity write $M = M(f)$.

Theorem. *If f has at least two distinct zeros, then*

$$z < cM^{3n} \log^3 |2b|,$$

where c is an effectively computable constant depending only on n.

Remarks. If f is an irreducible monic and $b = 1$ then this inequality was proved by Brindza, Győry and Evertse with different constants (see [BEGy], Th. 4). Moreover, if $n > 2$ and f is irreducible then a profound result of Győry (cf. [Gy1] or [Gy2]) makes it possible to substitute cM^{3n} by an effective constant depending only on the discriminant of f.

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Auxiliary results

To prove our Theorem, we need two lemmas. In what follows, for any non-zero algebraic number α, $h(\alpha)$ and $H(\alpha)$ denotes the logarithmic height and the classical (ordinary) height of α, respectively.

Lemma 1. *Let $\mathbb{K}$ be an algebraic number field of degree n and denote by R and r the regulator and the unit rank of $\mathbb{K}$, respectively. There exists a fundamental sytem of units $\varepsilon_1, \dots, \varepsilon_r$ for $\mathbb{K}$ so that*

$$h(\varepsilon_i) \le c^* R \ , \quad i = 1, \dots, r$$

where c^* is an effectively computable constant depending only on n.

Proof. This statement is a consequence of Lemma 1 in [BGy]. For other versions of this result cf. [B2] or [H]. □

Lemma 2. *Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers and let $A_1, \dots, A_n$ be positive real numbers with $A_i \ge \max\{H(\alpha_i), e\}$ for $i = 1, \dots, n$. Furthermore, let $b_1, \dots, b_n$ be rational integers with $\alpha_1^{b_1} \dots \alpha_n^{b_n} \ne 1$ and suppose that B is a positive real number satisfying $B \ge \max_{i=1,\dots,n} |b_i|$ and $B \ge e$. Now we have*

$$|\alpha_1^{b_1} \dots \alpha_n^{b_n} - 1| \ge B^{-c' \log A_1 \dots \log A_n} \ ,$$

where c' is an effectively computable constant depending only on n and on the degree of $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ over $\mathbb{Q}$.

Proof. This is Theorem 1.2 in [PW]. □

Proof of the Theorem

We have two cases to distinguish.

First we assume that f has an irreducible factor $P \in \mathbb{Z}[x]$ of degree $t \ge 2$. Let α be a zero of P, moreover, let R, h, D and r be the regulator, class number, discriminant and unit rank of the field $\mathbb{K} = \mathbb{Q}(\alpha)$, respectively. In the sequel $c_1, c_2, \dots$ will denote effectively computable positive constants depending only on n. The well-known inequalities

$$hR \le \sqrt{|D|}(\log |D|)^{n-1}, \quad \text{(cf. e.g. [L])}$$

and

$$|D| \le n^n M(P)^{2n-2} \le n^n M^{2n-2} \quad \text{(cf. [M])}$$

imply

$$hR < c_1 M^n. \tag{2}$$

Let a denote the leading coefficient of f and $\beta_1, \dots, \beta_n$ be the zeros of $g(x) = a^{n-1} f(\frac{x}{a})$. Set

$$\Delta(g) = \prod_{\beta_i \ne \beta_j} (\beta_i - \beta_j)^2 \ ,$$

and write g in the form $g(x) = P_1^{k_1}(x)P_2(x)$ where P_1 and P_2 are relatively prime polynomials in $\mathbb{Z}[x]$ and P_1 is an irreducible monic of degree t; (actually $P_1(x) = a^t P(\frac{x}{a})$). Let $\beta_1, \ldots, \beta_t$ be the zeros of P_1 and (x, y) be an arbitrary, however, fixed solution to (1). The g.c.d. of the principal ideals $\langle ax - \beta_1\rangle$ and $\langle g(ax)(ax - \beta_1)^{-k_1}\rangle$ divides $\Delta^n(g)$, therefore, there are integral ideals A, B, C in $\mathbb{K}$ so that

$$A\langle ax - \beta_1\rangle = BC^w \quad \text{where } w = \frac{z}{(z, k_1)}, \tag{3}$$

furthermore,

$$\max\{N_{\mathbb{K}/\mathbb{Q}}(A), N_{\mathbb{K}/\mathbb{Q}}(B)\} \leq |a \cdot b \cdot \Delta(g)|^{n^2}.$$

Hence, by a well-known inequality (cf. for example [Gy3], Lemma 3) and by (2), the ideals A^h and B^h have generators α and β, respectively, with

$$\max\{\overline{|\alpha|}, \overline{|\beta|}\} \leq \exp(c_2 M^{n-1}(\log M)^n \log|2b|).$$

The relation (3) can be written as

$$\alpha(ax - \beta_1)^h = \varepsilon\beta\gamma^w$$

where γ is a generator of C^h and ε is a unit. Let $\varepsilon_1, \ldots, \varepsilon_r$ be a fundamental system of units for $\mathbb{K}$ satisfying Lemma 1. Then we can express ε as $\varepsilon = \rho\varepsilon_1^{l_1} \ldots \varepsilon_r^{l_r}$ where ρ is a root of unity and we may assume that $\max\limits_{1\leq i\leq r} |l_i| < w$ (the remaining factors, if any, are incorporated in γ).

If $|ax| \leq M(g) + 1$ then

$$2^z \leq |y|^z \leq (2M(g) + 1)^n$$

and the Theorem is proved. Otherwise, $|ax| > M(g) + 1$ and $|ax - \beta_i| > 1$, $i = 1, \ldots, n$ implies

$$|ax - \beta_i| \leq |a^{n-1}by^z|, \quad i = 1, \ldots, n,$$

$$|a^{n-1}by^z|^h \geq \max_{1\leq i\leq t} |ax - \beta_i|^h \geq \overline{|\varepsilon_1|}^{-nw} \ldots \overline{|\varepsilon_r|}^{-nw}\overline{|\alpha|}^{-n}\overline{|\beta|}^{-n}\overline{|\gamma|}^{w}$$

and

$$\overline{|\gamma|} \leq |a^{n-1}b|^{\frac{h}{w}}|y|^{nh}\overline{|\alpha|}^{\frac{n}{w}}\overline{|\beta|}^{\frac{n}{w}}\prod_{i=1}^{r}\overline{|\varepsilon_i|}^{n}.$$

If $w < nh$ then by $0.056 < R$ (cf. [Z]) we obtain $w < 20nhR$ and

$$z < c_3 M^{n-1}(\log(2M))^{n-1}.$$

In case of $w \geq nh$

$$\overline{|\gamma|} \leq M|b|^{\frac{1}{n}}|y|^{nh}\overline{|\alpha|}\,\overline{|\beta|}\prod_{i=1}^{r}\overline{|\varepsilon_i|}^{n},$$

and we get

$$\log H\left(\frac{\gamma}{\gamma^{(2)}}\right) \leq c_4 \log|2b| M^{n-1}(\log(2M))^n \log|y|\ .$$

We may assume that $|ax| \geq \frac{1}{2}|y|^{\frac{z}{n}}$. Indeed, otherwise $\max\limits_{1\leq i\leq n} |ax-\beta_i| \geq |y|^{\frac{z}{n}}$ yields

$$|ax| \geq |y|^{\frac{z}{n}} - M(g)$$

and the Theorem is proved. Supposing

$$\frac{|\beta_i-\beta_j|}{|ax-\beta_i|} \geq \frac{|\beta_2-\beta_1|}{|ax-\beta_2|},\quad 1\leq i,j\leq t,\ i\neq j$$

we have

$$\prod_{\substack{1\leq i,j\leq t\\ \beta_i\neq\beta_j}} \frac{|\beta_i-\beta_j|}{|ax-\beta_i|} \leq \frac{|\Delta(g)|\cdot 2^n}{|y|^z}.$$

Then

$$\frac{|\beta_2-\beta_1|}{|ax-\beta_2|} \leq |y|^{-\frac{z}{4}},$$

or else we can derive a bound for z better than stated in the Theorem. Avoiding the trivial case $\left(\frac{ax-\beta_1}{ax-\beta_2}\right)^h = 1$, whenever $\frac{1}{2}|y|^{\frac{z}{n}} \leq |\Delta(g)|^{n^2}$ we obtain

$$\log\left|\left(\frac{ax-\beta_1}{ax-\beta_2}\right)^h - 1\right| \leq \log\left(h\left|\frac{ax-\beta_1}{ax-\beta_2}-1\right|\right) \leq -\frac{z}{8}\log|y|.$$

Finally, Lemma 2 yields

$$0\neq\left|\left(\frac{ax-\beta_1}{ax-\beta_2}\right)^h - 1\right| = \left|\left(\frac{\varepsilon_1}{\varepsilon_1^{(2)}}\right)^{l_1h}\cdots\left(\frac{\varepsilon_r}{\varepsilon_r^{(2)}}\right)^{l_rh}\frac{\beta/\alpha}{\beta^{(2)}/\alpha^{(2)}}\left(\frac{\gamma}{\gamma^{(2)}}\right)^{wh} - 1\right| \geq$$
$$\geq \exp\left(-c_5\log|2b|M^{3n-3}(\log|2M|)^{3n-1}\log|y|\log w\right)$$

and the comparision of the upper and lower bounds completes the proof (in the first case).

In the easier second case all the zeros of g are integral. Let k_i denote the multiplicities of β_i, $i=1,2$.

Repeating the argument one can have

$$u_i(ax-\beta_i) = v_i y_i^w$$

where $w = \frac{z}{(a,k_1k_2)}$ and $u_i, v_i, y_i \in \mathbb{Z}$, $|y_i|>1,\ i=1,2$.

To derive a bound for w from the equation

$$Ay_1^w - By_2^w = C$$

$(A=u_2v_1,\ B=u_1v_2,\ C=u_1u_2(\beta_2-\beta_1))$ one can apply Lemma 2 again, and we have

$$\frac{z}{\log z} \leq c_6 \log M \log|2b|\ ,$$

and the Theorem is proved. $\square$

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