Handout I - Mathematics II

The aim of this handout is to briefly summarize the most important definitions and theorems, and to provide some sample exercises. The topics are discussed in detail at the lectures and seminars. Students should also consult the suggested readings.

1. Derivatives of real functions

1.1. Theory.

Definition 1.1. *Let $f : D \to \mathbb{R}$ be a function, where D is an interval, and let $x_0 \in D$. If the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists then we say that f is differentiable at x_0. In this case we write

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

for the derivate of f at x_0. If f is differentiable for all $x_0 \in D$ then we say that f is differentiable. We write f' for the derivative of f.

Theorem 1.1. *Suppose that $f(x)$ and $g(x)$ are differentiable functions for all $x \in \mathbb{R}$. Then we have*

$$(f(x) + g(x))' = f'(x) + g'(x), \quad (f(x) - g(x))' = f'(x) - g'(x),$$

$$(cf(x))' = cf'(x) \ (c \in \mathbb{R}), \quad (f(x)g(x))' = f'(x)g(x) + f(x)g'(x),$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}, \ (f(g(x)))' = f'(g(x)) \cdot g'(x).$$

Theorem 1.2. *All the elementary functions are differentiable. Further, we have*

$$c' = 0 \ (c \in \mathbb{R}), \ (x^\alpha)' = \alpha x^{\alpha-1} \ (\alpha \in \mathbb{R}),$$

$$(a^x)' = a^x \ln(a) \ (a > 0), \ (e^x)' = e^x,$$

$$(\log_a(x))' = \frac{1}{x \ln(a)}, \ (\ln(x))' = \frac{1}{x},$$

$$(\sin(x))' = \cos(x), \ (\cos(x))' = -\sin(x),$$

$$(\mathrm{ctg}(x))' = -\frac{1}{\sin^2(x)}, \ (\mathrm{tg}(x))' = \frac{1}{\cos^2(x)}, \ (\mathrm{arctg}(x))' = \frac{1}{1 + x^2}.$$

Here $\mathrm{arctg}(x)$ *is the inverse of the function* $\mathrm{tg}(x) : (-\pi/2, \pi/2) \to \mathbb{R}$.

1.2. **Sample exercises.**

Exercise 1.1. *Differentiate the following functions:*

$$2(x^5 - \sqrt{x}), \quad e^x sin(x), \quad \frac{\sqrt[5]{x} - \cos x}{x^2 + \ln x}, \quad \ln(\sin(x)).$$

Exercise 1.2. *Differentiate the following functions:*

$$\left(\frac{2^x \cos(\cos(x^2)) - x^2 + x}{\ln(\sqrt{x}) + \sqrt{\ln(x)}} \right)^3, \quad e^{x^2 \sin(3x)}.$$

Throughout this material all functions f are assumed to be **elementary functions**, with $f : D \to \mathbb{R}$, where D is an **interval**.

2. Analysis of real functions

2.1. **Theory.**

Definition 2.1. *Let f be a function and $x_0 \in D$. If there exists an $\varepsilon > 0$ such that for all $x \in D \cap (x_0 - \varepsilon, x_0 + \varepsilon)$ we have*

- *$f(x_0) \leq f(x)$, then f has a local minimum at x_0,*
- *$f(x_0) < f(x)$, then f has a strict local minimum at x_0,*
- *$f(x_0) \geq f(x)$, then f has a local maximum at x_0,*
- *$f(x_0) > f(x)$, then f has a strict local maximum at x_0.*

In any of the above cases we say that f has a local extremum at x_0.

Definition 2.2. *Let f be a function. If for all $x, y \in D$ with $x < y$ we have $f(x) \leq f(y)$, then we say that f is monotone increasing.*

If for all $x, y \in D$ with $x < y$ we have $f(x) \geq f(y)$, then we say that f is monotone decreasing.

Definition 2.3. *Let f be a function. If for all $x, y \in D$ and $\lambda \in [0, 1]$ we have $\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y)$, then we say that f is convex on D. If $-f$ is convex on D, then we say that f is concave on D.*

Let $x_0 \in D$. If for some $\varepsilon > 0$, f is convex on $(x_0 - \varepsilon, x_0)$ and concave on $(x_0, x_0 + \varepsilon)$, or vice versa, then x_0 is called an inflection point of f.

Theorem 2.1. *Suppose that f has local extremum at some $x_0 \in D$. Then we have $f'(x_0) = 0$.*

Theorem 2.2. *Suppose that $x_0 \in D$, and $f'(x_0) = f''(x_0) = \cdots = f^{(n)}(x_0) = 0$, but $f^{(n+1)}(x_0) \neq 0$. If $n + 1$ is even, then f has a local extremum at x_0. Further, if $f^{(n+1)}(x_0) < 0$ then f has a local maximum at x_0, and if $f^{(n+1)}(x_0) > 0$ then f has a local minimum at x_0, respectively. On the other hand, if $n + 1$ is odd, then f has no local extremum at x_0.*

Theorem 2.3. *Suppose that $f'(x) \geq 0$ for all $x \in D$. Then f is monotone increasing on D.*

Theorem 2.4. *Suppose that $f'(x) \leq 0$ for all $x \in D$. Then f is monotone decreasing on D.*

Theorem 2.5. *Suppose that $f''(x) \geq 0$ for all $x \in D$. Then f is convex on D.*

Theorem 2.6. *Suppose that $f''(x) \leq 0$ for all $x \in D$. Then f is concave on D.*

2.2. **Sample exercise.**

Exercise 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - 3x$. Give the complete analysis of f, that is: determine the zeroes of f; calculate the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$; determine the local extrema of f; determine the intervals where f is monotone; determine the intervals where f is convex/concave, and give the inflection points of f.*

3. L'Hospital's rule

3.1. **Theory.**

Theorem 3.1. *Let f, g be two functions, such that $g(x) \neq 0$ $(x \in D)$. Further, assume that either $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$, or $\lim_{x\to x_0} f(x) = \pm\infty$, $\lim_{x\to x_0} g(x) = \pm\infty$, where $x_0 \in D$, or x_0 is an endpoint of D (possibly $x_0 = \pm\infty$). Then we have*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)},$$

if the latter limit exists.

3.2. **Sample exercises.**

Exercise 3.1. *Calculate the following limits:*

$$\lim_{x\to 0} \frac{\sin x}{x}, \quad \lim_{x\to\infty} \frac{x}{e^x}, \quad \lim_{x\to 0} x \ln x.$$

4. Taylor polynomial, Taylor series

4.1. **Theory.**

Definition 4.1. *Let $f : [a, b] \to \mathbb{R}$ be a function differentiable n times. Then the Taylor polynomial of order n of $f(x)$ around a point $x_0 \in (a, b)$ is given by*

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

If $x_0 = 0$ then the above polynomial is called the MacLaurin polynomial of $f(x)$ of order n.

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a function differentiable $n+1$ times. Then we have*

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},$$

where ξ is a number between x and x_0, depending on x.

Definition 4.2. *Let $f : [a,b] \to \mathbb{R}$ be a function differentiable infinitely many times. Then the Taylor series of $f(x)$ around a point $x_0 \in (a,b)$ is given by*

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

If $x_0 = 0$ then the above series is called the MacLaurin series of $f(x)$ of order n.

4.2. **Sample exercises.**

Exercise 4.1. *Give the Taylor-series of the functions*

$$x^3 + 2x^2 - x + 3, \quad e^x, \quad \sin x, \quad \cos x.$$

5. Primitive functions, indefinite integral

5.1. **Theory.**

Definition 5.1. *Suppose that we have $F'(x) = f(x)$. Then we say that $F(x)$ is a primitive function of $f(x)$.*

Theorem 5.1. *Let $F(x)$ be a primitive function of $f(x)$. Then $G(x)$ is a primitive function of $f(x)$ if and only if $G(x) = F(x) + c$ is valid with some $c \in \mathbb{R}$.*

Definition 5.2. *The set of primitive functions of $f(x)$ is called the indefinite integral of $f(x)$. Notation: $\int f(x)dx = F(x) + c$ $(c \in \mathbb{R})$, where $F(x)$ is a primitive function of $f(x)$.*

Theorem 5.2. *We have*

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

and

$$\int \lambda f(x)dx = \lambda \int f(x)dx \ (\lambda \in \mathbb{R}).$$

Theorem 5.3. *We have*

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c \ (\alpha \neq -1), \quad \int \frac{1}{x}dx = \ln x + c,$$

$$\int \cos x dx = \sin x + c, \quad \int \sin x dx = -\cos x + c,$$

$$\int e^x dx = e^x + c, \quad \int \frac{1}{1+x^2}dx = \mathrm{arctg} x + c.$$

Theorem 5.4. *We have*

$$\int f'(x)f(x)^{\alpha}dx = \frac{f(x)^{\alpha+1}}{\alpha+1} + c \ (\alpha \neq -1)$$

and

$$\frac{f'(x)}{f(x)}dx = lnf(x) + c.$$

Theorem 5.5. *(Partial integration (integration by parts)) We have*

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx.$$

Theorem 5.6. *(Partial fractions) Suppose that $x^2 + ax + b$ has two distinct real roots, α and β. Then there exist real numbers A and B such that*

$$\frac{1}{x^2+ax+b} = \frac{1}{(x-\alpha)(x-\beta)} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\beta)}.$$

Thus we have

$$\int \frac{1}{x^2+ax+b}dx = A\ln(x-\alpha) + B\ln(x-\beta) + c.$$

Theorem 5.7. *(Integral with substitution) We have*

$$\int f(x)dx = \left(\int f(g(t))g'(t)dt\right)\left(g^{-1}(x)\right),$$

where $g(x)$ is an invertible function.

5.2. **Sample exercises.**

Exercise 5.1. *Determine the following indefinite integrals:*

$$\int \left(\sqrt{x} + \frac{1}{x}\right)dx, \quad \int (e^x + 2\cos x)dx, \quad \int \frac{3}{1+x^2}dx.$$

Exercise 5.2. *Determine the following indefinite integrals:*

$$\int \cos x \sin^3 x dx, \quad \int \frac{x}{(1+x^2)^4}dx, \quad \int \mathrm{ctg} x dx, \quad \int \frac{e^x}{2-e^x}.$$

Exercise 5.3. *Determine the following indefinite integrals:*

$$\int x\cos x dx, \quad \int x \ln x dx, \quad \int x^3 \sin x dx.$$

Exercise 5.4. *Determine the following indefinite integrals:*

$$\int \frac{x+1}{x^2+1}, \quad \int \frac{x+2}{x^2-1}dx, \quad \int \frac{2x-1}{x^2-x-2}dx.$$

Exercise 5.5. *Determine the following indefinite integrals:*

$$\int \frac{1}{\sqrt{x+2} + (\sqrt{x+2})^3}, \quad \int \frac{e^{2x}}{e^x-1}dx, \quad \int \sqrt{x}e^{\sqrt{x}}dx.$$

6. Riemann (definite) integral

6.1. Theory.

Definition 6.1. *Suppose that f is continuous on $D = [a, b]$, with $a, b \in \mathbb{R}$, $a < b$. For any positive integer n put $\delta_n := \frac{b-a}{2^n}$ and $x_i^{(n)} := a + i\delta_n$ for $i = 0, 1, \ldots, 2^n$. Further, let $t_i^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$ such that $f(t_i^{(n)}) \geq f(x)$ for all $x \in [x_i^{(n)}, x_{i+1}^{(n)}]$ $(i = 0, 1, \ldots, 2^n - 1)$. Put*

$$S_n := \sum_{i=0}^{2^n-1} f(t_i^{(n)})\delta_n.$$

Then the Riemann integral of $f(x)$ between a and b is defined as

$$\int_a^b f(x)dx = \lim_{n\to\infty} S_n.$$

Definition 6.2. *We use the following notation:*

$$\int_b^a f(x)dx = -\int_a^b f(x)dx, \quad \int_a^a f(x)dx = 0.$$

Theorem 6.1. *We have*

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

and

$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx \ (\lambda \in \mathbb{R}).$$

Theorem 6.2. *(Newton-Leibniz formula) Let $F(x)$ be a primitive function of $f(x)$. Then we have*

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

Theorem 6.3. *(Partial integration (integration by parts) for definite integrals) We have*

$$\int_a^b f'(x)g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx.$$

Theorem 6.4. *(Integral with substitution for definite integrals) We have*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt,$$

where $g(x)$ is an invertible function, and $\alpha = g^{-1}(a)$ and $\beta = g^{-1}(b)$.

6.2. Sample exercises.

Exercise 6.1. *Calculate the following integrals:*

$$\int_1^2 \left(\sqrt{x} + \frac{1}{x}\right) dx, \quad \int_0^\pi (e^x + 2\cos x)dx, \quad \int_{-1}^1 \frac{3}{1+x^2}dx.$$

Exercise 6.2. *Calculate the following integrals:*

$$\int_0^\pi \cos x \sin^3 x dx, \quad \int_{-2}^2 \frac{x}{(1+x^2)^4}dx, \quad \int_{\pi/4}^{\pi/2} \mathrm{ctg} x dx, \quad \int_0^{100} \frac{e^x}{2-e^x}.$$

Exercise 6.3. *Calculate the following integrals:*

$$\int_0^{2\pi} x\cos x dx, \quad \int_1^3 xe^x dx, \quad \int_0^{\pi/2} x^3 \sin x dx.$$

Exercise 6.4. *Calculate the following integrals:*

$$\int_{-10}^{10} \frac{x+1}{x^2+1}, \quad \int_1^4 \frac{x+2}{x^2-1}dx, \quad \int_5^8 \frac{2x-1}{x^2-x-2}dx.$$

Exercise 6.5. *Calculate the following integrals:*

$$\int_1^2 \frac{1}{\sqrt{x+2} + (\sqrt{x+2})^3}, \quad \int_1^2 \frac{e^{2x}}{e^x - 1}dx, \quad \int_1^2 \sqrt{x}e^{\sqrt{x}}dx.$$

7. Improprius (improper) integral

7.1. Theory.

Definition 7.1. *Let f be bounded on the interval $[a, \infty)$. Then the improprius integral of $f(x)$ between a and ∞ is defined by*

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx,$$

if the limit exists.

Definition 7.2. *Let f be bounded on the interval $(-\infty, b]$. Then the improprius integral of $f(x)$ between $-\infty$ and b is defined by*

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)dx,$$

if the limit exists.

Definition 7.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be bounded. Then the improprius integral of $f(x)$ between $-\infty$ and ∞ is defined by*

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

with an arbitrary $c \in \mathbb{R}$, if the latter improprius integrals exist.

Definition 7.4. *Let $f[a, b] \to \mathbb{R}$ such that f is bounded on $[a, c]$ for any c with $a < c < b$. Then the improprius integral of $f(x)$ between a and b is defined by*

$$\int_{a}^{b} f(x)dx = \lim_{c\to b} \int_{a}^{c} f(x)dx,$$

if the limit exists.

Definition 7.5. *Let $f : [a, b] \to \mathbb{R}$ such that f is bounded on $[c, b]$ for any c with $a < c < b$. Then the improprius integral of $f(x)$ between a and b is defined by*

$$\int_{a}^{b} f(x)dx = \lim_{c\to a} \int_{c}^{b} f(x)dx,$$

if the limit exists.

7.2. **Sample exercises.**

Exercise 7.1. *Calculate the following improprius integrals:*

$$\int_{1}^{\infty} \frac{2}{x^3}dx, \quad \int_{0}^{1} \frac{3}{\sqrt{x}}dx.$$

8. Applications of the integral

8.1. **Theory.**

Theorem 8.1. *Let $f, g : [a, b] \to \mathbb{R}$ be two functions, such that $f(x) \geq g(x)$ for $a \leq x \leq b$. Then the area of the domain enclosed by $f(x)$ and $g(x)$ over $[a, b]$ is given by*

$$\int_a^b (f(x) - g(x))dx.$$

Theorem 8.2. *Let T be a domain in R^3 (i.e. a solid), and let $A(x)$ denote the area of the intersection of T and the plain $\{(u, v, w) : u = x\}$ for $x \in \mathbb{R}$. Then the volume of T is given by*

$$V(T) = \int_{-\infty}^{\infty} A(x)dx,$$

if the above integral exists.

Theorem 8.3. *Let $f : [a, b] \to \mathbb{R}$ be a function. Then the length of the curve of $f(x)$ over $[a, b]$ is given by*

$$L(f) = \int_a^b \sqrt{1 + (f'(x))^2}dx$$

if the integral exists.

Theorem 8.4. *Let $f : [a, b] \to \mathbb{R}$ be a function and let T be the domain (solid) obtained by rotating the curve of f around he interval $[a, b]$ in R^3. Then the volume of T and the surface of T are given by*

$$V(T) = \int_a^b \pi f^2(x)dx$$

and

$$S(T) = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}dx$$

respectively, if the integrals exist.

Definition 8.1. *Let $f : [0, 2\pi] \to \mathbb{R}$ be a function. Then the Fourier-series of $f(x)$ is given by*

$$a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

with

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

and

$$a_k = \frac{1}{\pi} \int\limits_0^{2\pi} f(x) \cos kx dx, \quad b_k = \frac{1}{\pi} \int\limits_0^{2\pi} f(x) \sin kx dx \quad (k = 1, 2, \dots).$$

8.2. **Sample exercises.**

Exercise 8.1. *Calculate the area enclosed by the curves of the functions* $f(x) = x^2$ *and* $g(x) = x + 2$ *over the interval* $[-1, 2]$.

Exercise 8.2. *Calculate the length of the curve of* $f(x) = 2x + 3$ *over the interval* $[0, 3]$*, and the volume and surface of the solid* T *obtained by rotating* $f(x)$ *over this interval.*

Exercise 8.3. *Give the Fourier transform of the function* $f(x) = x$.