Handout II - Mathematics II

The aim of this handout is to briefly summarize the most important definitions and theorems, and to provide some sample exercises. The topics are discussed in detail at the lectures and seminars. Students should also consult the suggested readings.

1. Introduction to differential equations

1.1. Theory.

Definition 1.1. *An equation involving (possibly higher) derivatives of an unknown real function $y(x)$ together with its variable x, is called a differential equation. The order of equation is n, where $y^{(n)(x)}$ is the highest derivative of $y(x)$ occurring in the equation. In many cases we simply write only y in place of $y(x)$.*

Remark 1.1. *An equation of the type $y' = f(x)$ is a differential equation of order one. Observe that here y is just a primitive function of $f(x)$. So if $F(x)$ is any primitive function of $f(x)$, then all solutions of the above differential equation is given by $y = F(x) + c$ $(c \in \mathbb{R})$.*

Definition 1.2. *A differential equation of order n together with the initial conditions $y^{(i)}(x_i) = y_i$ $(i = 0, 1, \ldots, n-1)$ is called an initial value problem.*

Theorem 1.1. *Under certain conditions, an initial value problem admits a unique solution $y(x)$.*

Remark 1.2. *Recall from physics that we have $s'(t) = v(t)$ and $s''(t) = v'(t) = a(t)$, where $s(t)$, $v(t)$ and $a(t)$ is the distance covered, the velocity and the acceleration of a moving object, respectively as a function of time t.*

Assume that when we start observing the object, it has already covered distance s_0, and has velocity v_0. Suppose that the acceleration $a(t) = a$ is constant. Then we can find the function $s(t)$ explicitly.

For this, observe that we have the following initial value problem:

$$s''(t) = a, \quad s'(0) = v_0, \quad s(0) = s_0.$$

From this a simple calculation yields

$$s(t) = \frac{a}{2}t^2 + v_0 t + s_0.$$

1.2. Sample exercises.

Exercise 1.1. *Find all solutions of the following differential equations:* $y' = x^2 + 1$, $y' = \sin x + \cos x$, $y' = 2e^x + 3$, $y'' = 1$.

Exercise 1.2. *Solve the following initial value problems: differential equations: $y' = x^2 + 1$ and $y(1) = 1$; $y' = e^x + x$ and $y(0) = 0$; $y'' = x^2 - x - 1$ and $y'(0) = 0$, $y(0) = 1$.*

Exercise 1.3. *We drive our car with a speed of 72 km/h. We see an accident ahead, and slam on the brake. How large constant deceleration do we need to stop the car in 100 m?*

2. Slope fields and numerical methods for solving differential equations

2.1. Theory.

Definition 2.1. *Consider an order one differential equation of the form*

$$y' = f(x, y)$$

with $f: I \times J \to \mathbb{R} \times \mathbb{R}$, where I and J are open intervals. A function $y: I^* \to \mathbb{R}$ is a solution of the above differential equation, if $I^* \subseteq I$, $y(I^*) \subseteq J$ and for all $x \in I$ we have

$$y'(x) = f(x, y(x)).$$

A solution is called maximal if any other solution can be obtained as its restriction.

Theorem 2.1. *Suppose that $f(x, y)$ is differentiable as a function of y for all $x \in I$, and the derivatives are continuous on $I \times J$. Then for any $(x_0, y_0) \in I \times J$ the initial value problem*

$$y' = f(x, y), \quad y(x_0) = y_0$$

admits a uniquely determined maximal solution $y: I^* \to \mathbb{R}$.

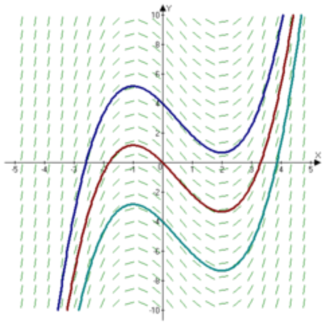
Remark 2.1. *When we give an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the solution curve (graph of the solution) has to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at some points (x, y). Each line segment has the same slope as the solution curve through (x, y), thus is tangent to the curve there. The resulting picture is called a slope field (or direction field) and gives a visualization of the general shape of the solution curves (see the figure below).*

Euler's method

Given a differential equation $y' = f(y, x)$ and an initial condition $y'(x_0) = y_0$, we can approximate the solution by its linearization

$$L(x) = y(x_0) + y'(x_0)(x - x_0) = y_0 + f(y_0, x_0)(x - x_0).$$

The function $L(x)$ gives a good approximation to the solution $y(x)$ in a short interval about x_0. The basis of Eulers method is to patch together a string of linearizations to approximate the curve over a longer stretch.



The method works in the following way. We know that the point (x_0, y_0) lies on the solution curve. Suppose that we specify a new value for the independent variable to be $x_1 = x_0 + dx$. If the increment dx is small, then

$$y_1 = L(x_1) = y_0 + f(y_0, x_0)dx$$

is a good approximation to the exact solution value $y = y(x_1)$. So from the point (x_0, y_0) which lies exactly on the solution curve, we have obtained the point (x_1, y_1) which lies very close to the point $(x_1, y(x_1))$ on the solution curve. Using the point (x_1, y_1) and the slope $f(x_1, y_1)$ of the solution curve through (x_1, y_1) we take a second step, setting $x_2 = x_1 + dx$, and so on. Continuing in this fashion, we get an approximation to one of the solutions by following the direction of the slope field of the differential equation.

Improved Euler's Method

We can improve on Eulers method by taking an average of two slopes. We first estimate y_n as in the original Euler method, but denote it by z_n. We then take the average of $f(x_{n-1}, y_{n-1})$ and $f(x_n, z_n)$ in place of $f(x_{n-1}, y_{n-1})$ in the next step. Thus, we calculate the next approximation y_n using

$$z_n = y_{n-1} + f(x_{n-1}, y_{n-1})dx$$
$$y_n = y_{n-1} + \frac{f(x_{n-1}, y_{n-1}) + f(x_n, z_n)}{2}dx.$$

Remark 2.2. *Euler's method have further variants and extensions, e.g. due to Runge and Kutta.*

2.2. **Sample exercises.**

Exercise 2.1. *Find the first three approximations y_1, y_2, y_3 using Eulers method for the initial value problem $y' = 1 + y$, $y(0) = 1$, starting at $x_0 = 0$ with $dx = 0.1$.*

Exercise 2.2. *Find the first three approximations y_1, y_2, y_3 using the extended Eulers method for the initial value problem $y' = 1 + y$, $y(0) = 1$, starting at $x_0 = 0$ with $dx = 0.1$. Compare the results with those obtained in the previous exercise.*

3. Separable differential equations

3.1. Theory.

Definition 3.1. *A differential equation of the form $y' = g(x)h(y)$ is called separable.*

Theorem 3.1. *The solutions of a separable differential equation $y' = g(x)h(y)$ can be obtained from the equation*

$$\int \frac{1}{h(y)} dy = \int g(x) dx + c$$

by expressing y in terms of x.

3.2. Sample exercises.

Exercise 3.1. *Solve the following differential equations:*

$$y' = xy, \quad y' = y + 1, \quad y' = \frac{\cos x}{y}.$$

Exercise 3.2. *Solve the following initial value problems:*

$$y' = x^2 y,\ y(1) = 0; \quad y' = \frac{x}{y},\ y(0) = 0.$$

4. Linear differential equations of order one

4.1. Theory.

Definition 4.1. *An equation of the form $y' + p(x)y = q(x)$ is called a linear differential equation of order one.*

Definition 4.2. *Let $y' + p(x)y = q(x)$ be a linear differential equation of order one. The equation $y' + p(x)y = 0$ is the homogenization of the original equation*

Theorem 4.1. *Let $y' + p(x)y = q(x)$ be a linear differential equation of order one. All solutions of the equation are of the form $y_c + y_p$, where y_c is the general (so-called complementary) solution of the homogenization of the equation, and y_p is a particular solution of the original equation.*

Solution of linear differential equations of order one. Consider a linear differential equation $y' + p(x)y = q(x)$ of order one. Observe that its homogenization $y' + p(x)y = 0$ is a separable differential equation. A simple calculation yields that its solutions are given by $y_c = y = cf(x)$ with $c \in \mathbb{R}$, where

$$f(x) = e^{-\int p(x)dx}.$$

In view of the previous theorem, now we only need to know a particular solution to the original equation. We find it by the method of the variation of the parameter. That is, we would like to find a solution of the form

$$y_p = y(x) = c(x)f(x).$$

For such a solution, we certainly have

$$y'(x) = c'(x)f(x) + c(x)f'(x) = c'(x)f(x) - c(x)f(x)p(x).$$

Substituting this assertion into the original equation, we get

$$c'(x)f(x) = q(x).$$

Hence we can easily find some $c(x)$, and then a particular solution $y_p = c(x)f(x)$. Then all solutions of the original equations are given by

$$y = y_c + y_p.$$

4.2. **Sample exercises.**

Exercise 4.1. *Solve the following differential equations:*

$$xy' - y = x^3 + 1.$$

Exercise 4.2. *Solve the following initial value problems:*

$$y' + xy = x^2, \;\; y(1) = 0.$$

5. Linear differential equations of order two

5.1. **Theory.**

Definition 5.1. *An equation of the form $P(x)y'' + Q(x)y' + R(x)y = G(x)$ is called a linear differential equation of order two. If $G(x)$ is identically zero, then the equation is called homogeneous, otherwise it is inhomogeneous.*

Solution of linear homogeneous differential equations of order two having constant coefficients. Consider a linear differential equation $ay'' + by' + cy = 0$ of order two. The quadratic equation

$$az^2 + bz + c = 0$$

is the so-called auxiliary equation of our differential equation. Let z_1, z_2 be the roots of the auxiliary equation.

If $b^2 - 4ac > 0$, that is z_1, z_2 are real roots, $z_1 \neq z_2$, then all solutions of the differential equation are given by

$$y = c_1 e^{z_1 x} + c_2 e^{z_2 x} \quad (c_1, c_2 \in \mathbb{R}).$$

If $b^2 - 4ac = 0$, that is z_1, z_2 are real roots, $z_1 = z_2$, then all solutions of the differential equation are given by

$$y = c_1 e^{z_1 x} + c_2 x e^{z_1 x} \quad (c_1, c_2 \in \mathbb{R}).$$

If $b^2 - 4ac < 0$, that is z_1, z_2 are so called complex roots, then we can write $z_1 = \alpha + i\beta$, $z_2 = \alpha - i\beta$, where $\alpha, \beta \in \mathbb{R}$ and $i = \sqrt{-1}$. Then all solutions of the differential equation are given by

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) \quad (c_1, c_2 \in \mathbb{R}).$$

Theorem 5.1. *Let $P(x)y'' + Q(x)y' + R(x)y = G(x)$ be a linear differential equation of order two. All solutions of the equation are of the form $y_c + y_p$, where y_c is the general (so-called complementary) solution of the homogenized equation $P(x)y'' + Q(x)y' + R(x)y = 0$, and y_p is a particular solution of the original equation.*

Solution of linear inhomogeneous differential equations of order two having constant coefficients. Consider a linear differential equation $ay'' + by' + cy = G(x)$ of order two. In view of the above theorem, we need to find y_c and y_p.

The functions y_c can be found by the method explained earlier.

To find y_p, we follow the method of undetermined coefficients. Namely:

• if $G(x)$ has a term which is a constant multiple of e^{rx} then include an expression

$$Ae^{rx}, \quad Axe^{rx}, \quad Ax^2e^{rx}$$

into y_p (with $A \in R$) according as r is not a root, a simple root or a double root of the auxiliary equation,

• if $G(x)$ has a term which is a constant multiple of $\sin(kx)$ or $\cos(kx)$, and k is not a root of the auxiliary equation, then include an expression

$$B\cos(kx) + C\sin(kx)$$

into y_p (with $B, C \in \mathbb{R}$),

• if $G(x)$ has a term which is a constant multiple of $px^2 + qx + m$, then include an expression

$$Dx^2 + Ex + F, \quad Dx^3 + Ex^2 + Fx, \quad Dx^4 + Ex^3 + Fx^2$$

into y_p (with $D, E, F \in R$) according as 0 is not a root, a simple root or a double root of the auxiliary equation.

5.2. **Sample exercises.**

Exercise 5.1. *Solve the following differential equations:*

$$y'' - y' - 6y = 0, \quad y'' + 4y' + 4 = 0, \quad y'' - 4y' + 5y = 0.$$

Exercise 5.2. *Solve the following initial value problem:*

$$y'' + 2y' + y = 0, \; y(0) = 1, \; y'(0) = 1.$$

Exercise 5.3. *Solve the following differential equation:*

$$y'' - y' = 5e^x - \sin(2x).$$

6. Laplace transform and its applications

6.1. **Theory.**

Definition 6.1. *Let $f : (0,\infty) \to \mathbb{R}$ be a function which is continuous up to finitely many points, and $f(t)e^{-at}$ is bounded for some positive real number a. Then the Laplace transform of f is defined as*

$$L[f(t)] = F(s) = \int_a^\infty f(t)e^{-st}dt.$$

The inverse Laplace transform is given by

$$L^{-1}[F(s)] = f(t).$$

The basic Laplace transforms are given in the following table.

$f(t)$	1	$e^{-at}t^n$ $(a \geq 0, n \geq 0)$	$\sin \omega t$	$\cos \omega t$
$F(s) = L[f(t)]$	$\frac{1}{s}$	$\frac{n!}{(s+a)^{n+1}}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{s}{s^2+\omega^2}$

Table 1. Basic Laplace transforms

Theorem 6.1. *The Laplace transform is linear, that is we have*

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$$

for any $a, b > 0$ and functions f, g as above.

Remark 6.1. *The Laplace transform is not multiplicative, that is, in general we have*

$$L[f(t)g(t)] \neq L[f(t)]L[g(t)]$$

for functions f, g as above.

Theorem 6.2. *(Basic Laplace transform operations) Let f be as above, and let $F(s) = L[f(t)]$. Then we have*

$$L[f'(t)] = sF(s) - f(0), \quad L[e^{-at}f(t)] = F(s+a) \ (a \geq 0),$$

and

$$f(0) = \lim_{s\to\infty} sF(s).$$

Application of Laplace transform for differential equations. Let b_0, b_1, b_2 be real numbers, and consider the differential equation

$$b_2y'' + b_1y' + b_0y = f(t),$$

assuming that for all functions appearing here we can apply Laplace transform. Then we have

$$L[b_2y'' + b_1y' + b_0y] = L[f(t)],$$

yielding

$$b_2L[y''] + b_1L[y'] + b_0L[y] = F(s),$$

where $F(s) = L[f(t)]$. Letting $Y(s) = L[y(t)]$, we get

$$b_2(s^2Y(s) - sy(0) - y'(0)) + b_1(sY(s) - y(0)) + b_0Y(s) = F(s).$$

This gives

$$Y(s) = \frac{F(s)}{b_2s^2 + b_1s + b_0} + \frac{sb_2y(0) + b_2y'(0) + b_1y(0)}{b_2s^2 + b_1s + b_0}.$$

From this, using inverse Laplace transform, we obtain y.

6.2. **Sample exercises.**

Exercise 6.1. *Calculate the Laplace transforms of the functions*

$$1, t, t^n, e^{-at}.$$

Exercise 6.2. *Solve the following initial value problems using Laplace transform:*
a) $y' + 2y = 12$, $y(0) = 10$,
b) $y' + 2y = 12\sin(4t)$, $y(0) = 10$,
c) $y'' + 3y' + 2y = 24$, $y(0) = 10$, $y'(0) = 0$,
d) $y'' + 2y' + 5y = 20$, $y(0) = 0$, $y'(0) = 10$.