PERFECT POWERS FROM PRODUCTS OF CONSECUTIVE TERMS IN ARITHMETIC PROGRESSION

K. GYŐRY[1], L. HAJDU[2], Á. PINTÉR[2]

Abstract. We prove that for any positive integers x, d, k with $\gcd(x, d) = 1$ and $3 < k < 35$ the product $x(x+d)\dots(x+(k-1)d)$ cannot be a perfect power. This yields a considerable extension of previous results of Győry, Hajdu and Saradha [15] and Bennett, Bruin, Győry and Hajdu [3] which covered the cases $k \leq 11$. We also establish more general theorems for the case when the product yields an almost perfect power. As in [15] and [3], for fixed k we reduce the problem to systems of ternary equations. However, our results do not follow as a mere computational sharpening of the approach utilized in [15] and [3], but instead require the introduction of fundamentally new ideas. For $k > 11$, a great number of new ternary equations arise that we solve by combining the Frey curve and Galois representation approach with local and cyclotomic considerations. Furthermore, the number of systems of equations grow so rapidly with k that, in contrast with the previous proofs, it is practically impossible to handle all cases one-by-one. The main novelty of this paper is that we algorithmize our proofs. We apply in a well-determined order an algorithm for solving some of the arising new ternary equations as well as several sieves based on the ternary equations already solved. This enables us to exclude by means of a computer the solvability of an enormous number of systems of equations under consideration. Our general algorithm seems to work for larger k as well, but there is of course a computational time limit.

1. Introduction and new results

A classical theorem of Erdős and Selfridge [12] says that the product of consecutive positive integers is never a perfect power. A natural

---

2000 *Mathematics Subject Classification.* 11D61, 11B25.

Key words and phrases. Perfect powers, arithmetic progression, modular forms.

1) Research supported in part by the Hungarian Academy of Sciences, and by the OTKA grant T67580.

2) Research supported in part by the Hungarian Academy of Sciences, and by the OTKA grants T48791 and T67580.

generalization is the Diophantine equation

$$x(x+d)\dots(x+(k-1)d) = by^n, \tag{1}$$

in non-zero integers x, d, k, b, y, n with $\gcd(x, d) = 1$, $d \geq 1$, $k \geq 3$, $n \geq 2$ and $P(b) \leq k$. Here $P(u)$ denotes the largest prime divisor of a non-zero integer u, with the convention that $P(\pm 1) = 1$.

Equation (1) has an extremely rich literature. For $d = 1$, equation (1) has been completely solved by Saradha [24] (for $k \geq 4$) and Győry [13] (for $k < 4$). Instead of trying to overview all branches of related results for $d > 1$ (which seems to be an enormous task), we refer to the excellent survey papers of Tijdeman [29] and Shorey [26], [27]. Here we mention only those contributions which are closely related to the results of the present paper, that is which provide the complete solution of (1) when the number k of terms is fixed.

If $(k, n) = (3, 2)$, equation (1) has infinitely many solutions even with $b = 1$. Euler (see [11]) showed that (1) has no solutions if $b = 1$ and $(k, n) = (3, 3)$ or $(4, 2)$. A similar result was obtained by Obláth [20], [21] for $(k, n) = (3, 4)$, $(3, 5)$ and $(5, 2)$. By a conjecture of Erdős equation (1) has no solutions in positive integers when $k > 3$ and $b = 1$. In other words, the product of k consecutive terms in a coprime positive arithmetic progression with $k > 3$ can never be a perfect power. By coprime positive progression we mean one of the form

$$x, x+d, \dots, x+(k-1)d,$$

where x, d are positive integers with $\gcd(x, d) = 1$.

Erdős' conjecture has recently been verified for certain values of k in a more general form. In the following Theorem A the case $k = 3$ is due to Győry [14], the cases $k = 4, 5$ to Győry, Hajdu, Saradha [15], and the cases $6 \leq k \leq 11$ to Bennett, Bruin, Győry, Hajdu [3].

Theorem A. *Suppose that k and n are integers with $3 \leq k \leq 11$, $n \geq 2$ prime and $(k, n) \neq (3, 2)$, and that x and d are coprime integers. If, further, b and y are non-zero integers with $P(b) \leq P_{k,n}$ where $P_{k,n}$ is given in Table 1, then the only solutions to (1) are with (x, d, k) in the following list:*

$$(-9,2,9), (-9,2,10), (-9,5,4), (-7,2,8), (-7,2,9),$$
$$(-6,1,6), (-6,5,4), (-5,2,6), (-4,1,4), (-4,3,3),$$
$$(-3,2,4), (-2,3,3), (1,1,4), (1,1,6).$$

It is a routine matter to extend Theorem A to arbitrary (that is, not necessarily prime) values of n. Further, we note that knowing the

k	$n=2$	$n=3$	$n=5$	$n \geq 7$
3	–	2	2	2
4	2	3	2	2
5	3	3	3	2
6	5	5	5	2
7	5	5	5	3
8	5	5	5	3
9	5	5	5	3
10	5	5	5	3
11	5	5	5	5

Table 1

values of the unknowns on the left-hand side of (1), one can easily determine all the solutions (x, d, k, b, y, n) of (1).

Very recently, for $k = 5, 6$ and $n \geq 7$ the bounds $P_{k,n}$ have been improved to 3 by Bennett [2]. Further, for $n = 2$ and positive x, Theorem A has been extended by Hirata-Kohno, Laishram, Shorey and Tijdeman [17]. In fact they did not handle (1) for some exceptional values of $b > 1$ for which (1) has been solved later by Tengely [28]. Putting together the results in [17] and [28], the following theorem holds.

Theorem B. *Equation* (1) *with* $n = 2$, $d > 1$, $5 \leq k \leq 100$ *and* $P(b) \leq k$ *has no solution in positive integer* x.

In case of $b = 1$, the assumption $k \leq 100$ can be replaced by $k \leq 109$ in Theorem B (see [17]). When $n = 3$, Hajdu, Tengely and Tijdeman [16] obtained the following extension of Theorem A.

Theorem C. *Suppose that* $n = 3$ *and that* (x, d, k, b, y) *is a solution to equation* (1) *with* $k < 32$ *such that* $P(b) \leq k$ *if* $4 \leq k \leq 12$ *and* $P(b) < k$ *if* $k = 3$ *or* $k \geq 13$. *Then* (x, d, k) *belongs to the following list:*

$$(-10, 3, 7), (-8, 3, 7), (-8, 3, 5), (-4, 3, 5), (-4, 3, 3), (-2, 3, 3),$$
$$(-9, 5, 4), (-6, 5, 4), (-16, 7, 5), (-12, 7, 5),$$
$$\textit{and } (x, 1, k) \textit{ with } -30 \leq x \leq -4 \textit{ or } 1 \leq x \leq 5,$$
$$(x, 2, k) \textit{ with } -29 \leq x \leq -3.$$

Further, if $b = 1$ *and* $k < 39$, *then we have*

$$(x, d, k, y) = (-4, 3, 3, 2), (-2, 3, 3, -2), (-9, 5, 4, 6), (-6, 5, 4, 6).$$

Theorems A, B and C confirm the conjecture of Erdős for the corresponding values of k and n. Moreover, under some additional assumptions on $P(b)$ they provide the complete solution of (1) for $b > 1$ as well.

In the present paper we considerably extend Theorem A, up to $k < 35$. Our main result is the following theorem which proves Erdős' conjecture for $k < 35$.

Theorem 1.1. *The product of k consecutive terms in a coprime positive arithmetic progression with $3 < k < 35$ is never a perfect power.*

For $n = 2$, $n = 3$ as well as for $k \le 11$, Theorem 1.1 follows from the above mentioned results. The remaining cases are covered by the following theorems.

Theorem 1.2. *Equation* (1) *has no solutions with $n \ge 7$ prime, $12 \le k < 35$ and $P(b) \le P_{k,n}$, where*

$$P_{k,n} = \begin{cases} 7, & \text{if } 12 \le k \le 22, \\ \frac{k-1}{2}, & \text{if } 22 < k < 35. \end{cases}$$

Theorem 1.3. *The only solutions to equation* (1) *with $n = 5$, $8 \le k < 35$ and $P(b) \le P_{k,5}$, with*

$$P_{k,5} = \begin{cases} 7, & \text{if } 8 \le k \le 22, \\ \frac{k-1}{2}, & \text{if } 22 < k < 35 \end{cases}$$

are given by

$$(k,d) = (8,1),\ x \in \{-10,-9,-8,1,2,3\}; \quad (k,d) = (8,2),\ x \in \{-9,-7,-5\};$$

$$(k,d) = (9,1),\ x \in \{-10,-9,1,2\}; \quad (k,d) = (9,2),\ x \in \{-9,-7\};$$

$$(k,d) = (10,1),\ x \in \{-10,1\}; \quad (k,d,x) = (10,2,-9).$$

Note that in the case $n = 3$ Theorem 1.3 yields an extension of Theorem A already for $8 \le k \le 11$.

Similarly as in [15] and [3], results on equation (1) have a simple consequence for the rational solutions of equations of the form

$$u(u+1)\dots(u+k-1) = v^n. \tag{2}$$

More precisely, we have the following

Corollary 1.1. *Suppose that $n \ge 2$, $1 < k < 35$ and $(k,n) \ne (2,2)$. Then equation* (2) *has no solutions in positive rational numbers u, v.*

For $k \leq 11$, this was proved in [15]. When $k > 11$, the statement is a straightforward consequence of Theorem 1.1, see [15] and [3] for the necessary arguments. We note that equation (2) has been first studied by Sander [23].

In the case $k \leq 11$ and $n \geq 5$, equation (1) was reduced in Győry [14], Győry, Hajdu and Saradha [15], Bennett, Bruin, Győry and Hajdu [3] and Bennett [2] to finitely many ternary equations of signature (n, n, n), $(n, n, 2)$ or $(n, n, 3)$. In our paper we follow the same strategy. However, for $k > 11$ and $n \geq 7$ prime, numerous new ternary equations of signature $(n, n, 2)$ arise which must be solved under certain arithmetic conditions. On solving these equations we combine the Frey curve and modular Galois representation approach with local methods and some classical work on cyclotomic fields. These results may be of independent interest. For the most part, our results concerning ternary equations do not follow from straightforward application of the modularity of Galois representations attached to Frey curves, it is also necessary to understand the reduction types of these curves at certain small primes.

For increasing k, the number of possible k-tuples $(a_0, \ldots, a_{k-1})$ introduced in (3) below and hence also the number of arising systems of ternary equations grow so rapidly with k, that in contrast with the cases $k \leq 11$ treated in [14], [15], [3], [2], practically it is already impossible to handle all cases one-by-one without using computer. The principal novelty of our paper is that we algorithmize our proof. For fixed k, we combine our algorithm for solving the new ternary equations with several sieves based on the arising ternary equations already solved, and we use a computer to exclude the solvability of enormous number of systems of ternary equations. Our general method seems to work for larger k as well, we do not see any theoretical obstacle to extend the results even further. However, the time consumption of the method increases rather rapidly, that is why we stopped at $k = 34$. As it can be of some interest, we give a few details here.

We have used a 2.4 MHz PC with a Quad processor to execute the calculations. To establish our new results for ternary equations of signature $(n, n, 2)$ (see Proposition 2.2) we have implemented our algorithm in Magma [7]. The total running time to prove Proposition 2.2 was about two weeks. The proof of Theorem 1.1 goes via proving Theorems 1.2 and 1.3. To verify the latter results, we have implemented our sieving procedures in Maple, separately for the cases $n \geq 7$ and $n = 5$. In both cases the running time of the program was the following: a few seconds up to $k = 19$, a few minutes up to $k = 23$, a few hours up to $k = 29$, a few days for $k = 30, 31$ and about a week for $k = 32, 33, 34$ each. Altogether, after having Proposition 2.2 the calculations to prove

Theorems 1.2 and 1.3 took about a month each. We mention that because of the extremely huge number of cases to be looked after, having only the "ternary" results it is hopeless to attack the problem without some additional, new "sieving" ideas. Vice versa, using only the sieving procedures with the previously known "ternary" results, one would be left with a lot of cases which are not handled. So to prove our results, we need to find a balanced and efficient combination of both techniques.

For $n = 5$, hardly any information is available through the theory of "general" modular forms. In this case we make use of some classical and new results concerning equations of the shape $AX^5 + BY^5 = CZ^5$.

The organization of the paper is as follows. In the next section we introduce notation and summarize some old and establish some new results about ternary equations which we use in the paper. The final section is devoted to the proofs of the theorems.

2. Notation and auxiliary results

For integers d, x, k and indices $0 \le i_1 < \ldots < i_l < k$ put

$$\Pi(i_1, \ldots, i_l) = (x + i_1 d) \ldots (x + i_l d)$$

and

$$\Pi_k = \Pi(0, 1, \ldots, k-1) = x(x+d) \ldots (x + (k-1)d).$$

Assume that (1) has a solution in non-zero integers x, d, k, b, y, n with the requested properties. From (1) one can then deduce that

$$x + id = a_i x_i^n \quad (i = 0, 1, \ldots, k-1) \tag{3}$$

where x_i is a non-zero integer and a_i is an nth power free integer with $P(a_i) \le k$. For given k, there are only finitely many and effectively determinable such k-tuples $(a_0, a_1, \ldots, a_{k-1})$.

For brevity, we introduce the following notation. Write

$$[i_1, i_2, i_3]: \quad c_{i_1} a_{i_1} x_{i_1}^n + c_{i_3} a_{i_3} x_{i_3}^n = c_{i_2} a_{i_2} x_{i_2}^n \tag{4}$$

where $0 \le i_1 < i_2 < i_3 < k$ and $c_{i_1} = (i_3 - i_2)/D, c_{i_2} = (i_3 - i_1)/D, c_{i_3} = (i_2 - i_1)/D$ with $D = \gcd(i_3 - i_2, i_3 - i_1, i_2 - i_1)$. Further, if $0 \le j_1 < j_2 \le j_3 < j_4 < k$ with $j_1 + j_4 = j_2 + j_3$, then let

$$[j_2, j_3] \times [j_1, j_4]: \quad a_{j_2} a_{j_3} (x_{j_2} x_{j_3})^n - a_{j_1} a_{j_4} (x_{j_1} x_{j_4})^n = (j_2 j_3 - j_1 j_4) d^2.$$

Given a k-tuple $(a_0, a_1, \ldots, a_{k-1})$, we obtain in this way a complicated system of ternary equations to be solved.

In the proofs of our theorems we use several results concerning ternary equations to solve the arising systems of equations. In this section we collect some earlier theorems and establish two new results for ternary equations which we need later on. We start with ternary equations of signature $(n, n, 2)$.

Proposition 2.1. *Let $n \geq 7$ be prime, u, v, w nonnegative integers, and A and B coprime non-zero integers. Then the following Diophantine equations have no solutions in pairwise coprime non-zero integers X, Y, Z with $XY \neq \pm 1$:*

$$X^n + 2^u Y^n = 3^v Z^2, \ u \neq 1 \tag{5}$$

$$X^n + Y^n = CZ^2, \ C \in \{2, 6\} \tag{6}$$

$$X^n + 5^u Y^n = 2Z^2 \text{ with } n \geq 11 \text{ if } u > 0 \tag{7}$$

$$AX^n + BY^n = Z^2, \ AB = 2^u p^v, \ u \neq 1, \ p \in \{11, 19\}. \tag{8}$$

Proof. This result is due to Bennett, Bruin, Győry and Hajdu [3]. □

The following result is new. For its formulation, we need a further standard notation. If m is a positive integer, let $\mathrm{rad}(m)$ denote the *radical* of m, i.e. the product of distinct prime divisors of m with the convention that $\mathrm{rad}(1) = 1$.

Set

$$\begin{aligned} I_1 = \{&(2,1),(2,3),(2,5),(2,7),(6,1),(6,5),(10,1),(10,3),(14,1),\\ &(14,3),(22,1),(26,1),(30,1),(34,1),(38,1),(42,1),(46,1),(66,1),\\ &(70,1),(78,1),(102,1),(114,1),(102,1),(114,1),(130,1),(138,1)\}, \end{aligned}$$

$$\begin{aligned} I_2 = \{&(3,1),(3,5),(5,1),(5,3),(7,1),(13,1),(15,1),(17,1),(21,1),(23,1),\\ &(33,1),(35,1),(39,1),(51,1),(57,1),(69,1),(165,1)\} \end{aligned}$$

and

$$\begin{aligned} I_3 = \{&(3,2),(5,6),(7,2),(11,2),(13,2),(15,2),(17,2),(19,2),(21,2),\\ &(23,2),(33,2),(35,2),(39,2)\}. \end{aligned}$$

Proposition 2.2. *Let $n > 31$ be a prime, A, B, C pairwise coprime positive integers with $(\mathrm{rad}(AB), C) \in I_1 \cup I_2 \cup I_3$ and $p \in \{11, 13, 17, 19, 23, 29, 31\}$ such that $p \nmid AB$. Then the equation*

$$AX^n + BY^n = CZ^2 \tag{9}$$

has no solutions in pairwise coprime non-zero integers X, Y, Z with $p \mid XY$, unless, possibly, in the cases listed in Table 2.

As we mentioned in the introduction, to prove our results in the case $n \geq 7$ we had to find an efficient combination of the "modular"

n	(rad(AB),C,p)
37	(2,7,31), (3,5,31), (6,5,31), (19,2,29), (22,1,31), (46,1,29), (46,1,31),
	(70,1,29)
41	(2,7,11), (21,2,13), (21,2,19), (21,2,29), (22,1,31), (46,1,31), (51,1,13),
	(102,1,13), (165,1,13), (165,1,31)
43	(5,6,13), (6,5,23)
47	(5,6,11), (5,6,29), (6,5,31), (15,2,11), (15,2,29), (33,2,13), (33,2,23), (39,2,31)
59	(3,5,31), (6,5,31), (39,2,23), (165,1,17)
61	(5,6,13), (5,6,29),(14,3,17), (15,2,13), (15,2,29), (39,2,17), (39,2,19)
67	(165,1,29)
71	(33,2,23)
79	(5,6,17), (15,2,17), (165,1,19)
83	(165,1,29)
89	(165,1,29), (165,1,31)
97	(5,6,31), (15,2,31), (165,1,29)
107	(5,6,31), (15,2,31)
127	(33,2,31), (165,1,29)
137	(5,6,23)
193	(5,6,31), (15,2,31)
229	(33,2,31)
239	(33,2,31), (165,1,29)

TABLE 2

and "sieving" techniques. A very great number of new ternary equations arose for each $k > 11$. We used the following strategy. We first solved a few well-chosen ternary equations (considering only a small subset I of $I_1 \cup I_2 \cup I_3$ in Proposition 2.2), and using our sieves (which will be detailed in the next section) we tried to reduce each case to ternary equations either treated already in Propositions 2.1, 2.4 or 2.5 or belonging to I. After a while (for larger values of k) there were exceptional cases where such a reduction was unavailable. At that point we enlarged the set I in several steps and gradually we reached the finite sets I_1, I_2, I_3 in Proposition 2.2. By utilizing all the equations occurring in Propositions 2.1, 2.4, 2.5 or corresponding to $I_1 \cup I_2 \cup I_3$ in Proposition 2.2 we were able to "cover" all cases. For the details we refer to the proof of Theorem 1.2.

Proof. To solve our equations of the form (9) we shall apply the modular approach. Specifically, to a putative nontrivial solution x, y of (9) we associate a Frey curve $E/\mathbb{Q}$, with the corresponding mod n Galois representation

$$\rho_n^E : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_n)$$

on the n-torsion $E[n]$ of E. This representation arises from a cuspidal newform $f = \sum_{r=1}^{\infty} c_r q^r$ of weight 2 and trivial Nebentypus character. For details, we refer to [5]. As usual, for a positive integer m let $\mathrm{rad}_2(m)$ denote the *2-free radical* of m, i.e. the product of distinct odd prime divisors of m, with the convention that $\mathrm{rad}_2(1) = 1$. It can be shown that the level N of the newform considered above is contained in $\{2^\alpha \cdot \mathrm{rad}_2(AB) \cdot \mathrm{rad}_2^2(C), \alpha = 0, 1, 2, 3, 5, 7\}$, $\{2^\alpha \cdot \mathrm{rad}_2(AB) \cdot \mathrm{rad}_2^2(C), \alpha = 1, 5\}$, or $\{256 \cdot \mathrm{rad}_2(AB) \cdot \mathrm{rad}_2^2(C)\}$, according as $(\mathrm{rad}(AB), C) \in I_1, I_2$ or I_3, respectively. The assumption that $p|xy$ for a prime p with $p \in \{11, 13, 17, 19, 23, 29, 31\}$ implies that if p is relatively prime to N then

$$Norm_{K_f/\mathbb{Q}}(c_p \pm (p+1)) \equiv 0 \pmod{n}, \tag{10}$$

where c_p is the pth Fourier coefficient of f, and K_f is the field generated by the Fourier coefficients of f. This means that if (10) does not hold, we arrive at a contradiction. For the recipes of this technique see [1] or [8].

We illustrate our approach in the case $(\mathrm{rad}(AB), C) = (38, 1)$. The corresponding levels are $19, 2\cdot 19, 4\cdot 19, 8\cdot 19, 32\cdot 19$ and $128\cdot 19$. Suppose that x, y, z is a solution of the corresponding equation (9) in non-zero pairwise coprime integers such that $p \mid xy$, where p is a prime with $11 \le p \le 31$. Using a simple Magma program, we calculate the Fourier coefficients c_p of the corresponding one-dimensional newforms f at the levels considered above. Then we have

$$n \mid (c_p - (p+1))(c_p + p + 1) =: B_p. \tag{11}$$

For the corresponding higher dimensional newforms f at the levels under consideration we use a stronger sieve. Let

$$A_m = \mathrm{Norm}_{K_f/\mathbb{Q}}(c_m^2 - (m+1)^2) \prod_{\substack{|a| < 2\sqrt{m} \\ a \text{ is even}}} \mathrm{Norm}_{K_f/\mathbb{Q}}(c_m - a)$$

for $m = 3, 5, 7$. Our method yields now that

$$n \mid \gcd(B_p, A_3, A_5, A_7). \tag{12}$$

Consequently, if for some prime p with $11 \le p \le 31$ (11) and (12) do not hold for any f in question, then in the case $(\mathrm{rad}(AB), C) = (38, 1)$ equation (9) has no solution in pairwise coprime non-zero integers x, y, z with $p \mid xy$.

Using the same arguments for each equation considered in Proposition 2.2, we infer that equation (9) may have a solution with the prescribed properties only in the cases listed in Table 2.

We note that the Hasse-Weil bound implies that $B_p \neq 0$. Further, for the pairs $(\mathrm{rad}(AB), C)$ and for the higher dimensional case we omit A_m from the stronger sieve if $A_m = 0$ or $m|ABC$. □

Remark. We can choose further primes m for making a more stronger sieve. For example, in the case $(\mathrm{rad}(AB), C) = (165, 1)$ we can apply the sieve $n|\gcd(B_p, A_7, A_{61}, A_{73})$ for higher dimensional forms and we can exclude the cases

$$(n,p) = (41,13),(41,31),(59,17),(67,29),(79,19),\\ (89,31),(97,29),(127,29),(239,29) \tag{13}$$

as well. However, to find such appropriate primes m involves a long computation. Since for our later purposes Table 2 and its refinement excluding the cases listed in (13) are already sufficient, we do not continue this procedure.

We use ternary equations of signature $(n, n, 3)$ via the following result of Bennett [2]. For a prime p and non-zero integer u, $\mathrm{ord}_p(u)$ denotes as usual the largest integer v for which $p^v \mid u$ holds.

Proposition 2.3. *If x and d are coprime non-zero integers, then the Diophantine equation*

$$x(x+d)(x+3d)(x+4d) = by^n \tag{14}$$

has no solutions in non-zero integers b, y and prime n with $n \geq 7$ and $P(b) \leq 3$.

Proof. The statement is a simple consequence of a recent result of Bennett [2]. However, for the sake of completeness we give the main steps of the proof.

Suppose to the contrary that x, d, b, y, n is a solution to (14) with $by \neq 0$. If $3 \nmid x(x+d)$ then using the notation (3) the identity $[1,3] \times [0,4]$ gives

$$a_1a_3(x_1x_3)^n - a_0a_4(x_0x_4)^n = 3d^2,$$

and we also have $\gcd(a_1a_3x_1x_3, a_0a_4x_0x_4) = 1$ and $P(a_0a_1a_3a_4) \leq 2$. As either $\mathrm{ord}_2(a_1a_3) = \mathrm{ord}_2(a_0a_4) = 0$, or $\mathrm{ord}_2(a_1a_3) = 0$ and $\mathrm{ord}_2(a_0a_4) \geq 2$ (or vice versa), the statement follows from (5) of Proposition 2.1 in this case.

Otherwise, if $3 \mid x(x+d)$ then the identity $(x+d)^2(x+4d) - x(x+3d)^2 = 4d^3$ yields

$$a_1^2a_4(x_1^2x_4)^n - a_0a_3^2(x_0x_3^2)^n = 4d^3.$$

After simplifying with a suitable power of 2, we get an equality either of the form

$$X^n + 3^vY^n = 2^uZ^3,\ u \geq 1,\ v \geq 3,\ \gcd(X, 3Y) = 1,$$

or of the shape

$$AX^n + BY^n = Z^3,\ AB = 2^u 3^v,\ u \geq 1,\ v \geq 3,\ \gcd(AX, BY) = 1.$$

However, using results from [2] about certain ternary equations of signature $(n, n, 3)$, the statement follows also in this case. □

We will also use results on equations of signature (n, n, n).

Proposition 2.4. *Let $n \geq 3$ and $u \geq 0$ be integers. Then the Diophantine equation*

$$X^n + Y^n = 2^u Z^n$$

has no solutions in pairwise coprime non-zero integers X, Y, Z with $XYZ \neq \pm 1$.

Proof. This result is essentially due to Wiles [30] (in case $u \mid n$), Darmon and Merel [9] (if $u \equiv 1 \pmod{n}$) and Ribet [22] (in the remaining cases for $n \geq 5$ prime); see also Győry [14]. □

Proposition 2.5. *Let $n \geq 5$, and let A, B be coprime positive integers with $AB = 2^u 3^v$ or $2^u 5^v$, where u and v are non-negative integers with $u \geq 4$. Then the equation*

$$AX^n + BY^n = Z^n \tag{15}$$

has no solutions in pairwise coprime non-zero integers X, Y and Z.

Proof. This is Lemma 13 in [25]. It has been proved by the method involving Frey curves and modular forms. □

For $n = 5$, most of the above assertions on ternary equations cannot be applied. Then we shall use the following results as well.

Proposition 2.6. *Let $n \geq 3$ be an integer. All the solutions of the equation*

$$x(x+1)\dots(x+k-1) = by^n \tag{16}$$

in positive integers x, k, b, y with $k \geq 8$ and $P(b) \leq 7$ are given by

$$k \in \{8, 9, 10\} \quad \textit{and} \quad x \in \{1, 2, \dots, p^{(k)} - k\}, \tag{17}$$

where $p^{(k)}$ denotes the least prime satisfying $p^{(k)} > k$.

Proof. It follows from a theorem of Saradha [24] that, in (16), $P(y) \leq k$. As was seen in Győry [13], we then get $x \in \{1, 2, \dots, p^{(k)} - k\}$, whence $p^{(k)} > x + k - 1$. Denote by $p_{(k)}$ the greatest prime with $p_{(k)} \leq k$. Then, for $k \geq 11$, $p_{(k)} \geq 11$. Further, by Chebyshev's theorem $p^{(k)} < 2p_{(k)}$. In view of $p_{(k)} \leq k$ we have $p_{(k)} \mid x(x+1)\dots(x+k-1)$. But it follows that $2p_{(k)} > x + k - 1$. Hence (16) and $P(b) \leq 7$ give $p_{(k)}^n \mid$

$x(x+1)\dots(x+k-1)$, which implies that $p_{(k)}^n \le x+k-1$. Hence we get $p_{(k)}^l \le 2p^{(k)}$, a contradiction.

It remains to treat the case $k \in \{8,9,10\}$. Then $p^{(k)} = 11$ and it is easy to check that the values k, x listed in (17) are the solutions of (16). □

Lemma 2.1. *Let $n = 5$. For $k = 5$, $P(b) \le 3$, and for $6 \le k \le 11$, $P(b) \le 5$, equation (1) has the only solution $(x,d,k) = (-5,2,6)$ with $d \ge 2$.*

Proof. This is a special case of Theorem 1.2 in [3]. □

Lemma 2.2. *Let $n = 5$. Suppose that x, d, y, b provides a solution to (1) with $P(b) \le 3$ and $k = 4$. Then either $(x,d) = (-3,2)$, or, up to symmetry, $(a_0,a_1,a_2,a_3) = (4,3,2,1)$ or $(9,4,1,6)$.*

Proof. This is Lemma 6.3 in [3]. □

Let C be a 5th power free positive integer with $P(C) \le 7$. Then we can write

$$C = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7^\delta \tag{18}$$

with non-negative integers $\alpha, \beta, \gamma, \delta$ not exceeding 4.

Proposition 2.7. *If the diophantine equation*

$$X^5 + Y^5 = CZ^5 \tag{19}$$

has solution in non-zero pairwise coprime integers X, Y and Z, then either
(i) $C = 2$, $X = Y = \pm 1$, or
(ii) $C = 7^\delta$ with $1 \le \delta \le 4$, $5 \mid XY$, $5 \nmid Z$ and Z is odd, or
(iii) $C \in \{2\cdot 3^2 \cdot 7^\delta, 2^2\cdot 3^4\cdot 7^\delta, 2^3\cdot 3\cdot 7^\delta\}$ with $1 \le \delta \le 4$ and $5 \mid Z$.

This implies that if in (19) $5 \nmid XYZ$, then (i) holds. If in particular $P(C) \le 5$, then Proposition 2.7 gives Proposition 6.1 of [3].

Proof. Let X, Y, Z be a solution of (19) in non-zero pairwise coprime integers. By results of Dirichlet and Dénes [10], it suffices to deal with the case $C > 2$ and $XYZ \ne \pm 1$. It follows from a theorem of Lebesgue ([11], p. 738, item 37) that $5 \nmid C$ and

$$C \equiv \pm 1, \pm 7 \pmod{5^2}. \tag{20}$$

First assume that $5 \nmid Z$. We have

$$C^4 \equiv 1 \pmod{5^2} \quad \text{and} \quad 2^4 \not\equiv 1 \pmod{5^2},$$

whence

$$C^4 \not\equiv 2^4 \pmod{5^2}.$$

Applying Lemmas 6.1 and 6.2 of [4] to (19), we deduce that $5 \mid XY$, CZ is odd and

$$r^4 \equiv 1 \pmod{5^2} \tag{21}$$

for each prime divisor r of C. In view of (18) and (21) we infer that only $r = 7$ can hold, and (ii) follows.

Now suppose that $5 \mid Z$. The prime 5 being regular, a theorem of Maillet (see e.g. [11], p. 759, item 167) implies that C must have at least three distinct prime factors. This means that in (18) $\gamma = 0$ and $\alpha, \beta, \delta \geq 1$. It is easy to check that together with (20) this gives (iii). □

3. Proofs

First we prove Theorems 1.2 and 1.3. As we mentioned already, we need to consider the cases $n = 5$ and $n \geq 7$ separately. The reason is that the theory of ternary equations cannot be efficiently applied in case of $n = 5$. We start with $n \geq 7$.

Proof of Theorem 1.2. To prove the theorem we eventually reduce the problem to the solution of several ternary diophantine equations. We start with explaining the main ideas. Suppose that under the assumptions of our theorem equation (1) has a solution. First observe that, by (3), to determine all solutions to (1) with fixed k it is sufficient to characterize the arithmetic progressions of the shape $a_0x_0^n, a_1x_1^n, \ldots, a_{k-1}x_{k-1}^n$ with the properties that $\gcd(a_0x_0^n, a_1x_1^n) = 1$ and

$$P(a_i) \leq k \text{ and } a_i \text{ is } n\text{th power free for } i = 0, 1, \ldots, k-1. \tag{22}$$

Further, the assumption $P(b) \leq P_{k,n}$ implies that

$$n \mid \mathrm{ord}_p\left(\prod_{i=0}^{k-1} a_i\right) \text{ for all primes } p > P_{k,n}. \tag{23}$$

In particular, if p is a prime and $u \geq 1$ is an integer with $p^u \mid a_ix_i^n$ then $p^u \mid a_jx_j^n$ if and only if $p^u \mid i - j$. This assertion will be used later on without any further reference. The number of possible k-tuples $(a_0, a_1, \ldots, a_{k-1})$ with properties (22) and (23) grows very rapidly with k, and it is impossible to look at them one-by-one if k is relatively large. So we apply the following strategy.

We exclude the possible coefficient k-tuples $(a_0, a_1, \ldots, a_{k-1})$ in several steps, using certain procedures in a well-determined order. A k-tuple will be excluded after assuring that in the corresponding case equation (1) cannot have other solutions than those listed in the statement. We start with arguments with which we can exclude a great number of k-tuples $(a_0, a_1, \dots, a_{k-1})$. By induction we can exclude a lot of possibilities. Namely, if for some $\ell \geq 3$ $P(a_0 \dots a_{\ell-1}) \leq P_{\ell,n}$ or $P(a_{k-\ell} \dots a_{k-1}) \leq P_{\ell,n}$ holds, then the statement follows either by induction, or by Theorem A. By this observation the number of cases to be considered can be reduced drastically. Subsequently, after each step, it will be simpler and simpler to manage and exclude the remaining k-tuples. We shall explain the details later on, at the sieves. Further, we provide examples to illustrate how the sieves work.

In what follows, we always assume that k is fixed with $11 < k < 35$. We use the following convention. Let $2 = p_1 < p_2 < \dots < p_{\pi(k-1)}$ be the primes $\leq k-1$, where $\pi(k-1)$ denotes the number of primes not exceeding $k-1$. Observe that as $P_{k,n} < k$ for $n \geq 7$, by (23) we have $P(a_i) < k$ in (22) for all $i = 0, 1, \dots, k-1$. We indicate the distribution of these primes among the a_i resp $a_i x_i^n$ (or in other words, the prime divisors of the a_i resp. $a_i x_i^n$) by the help of certain $\pi(k-1)$-tuples of the form $(m_{\pi(k-1)}, \dots, m_1)$. For $3 \leq j \leq \pi(k-1)$ let

$$m_j \in \{\times, 0, 1, \dots, p_j - 1\}$$

where $m_j = \times$ if $p_j \nmid \Pi_k$ (i.e. p_j does not divide $x(x+d)\dots(x+(k-1)d)$); otherwise, let m_j denote the integer from among $0, 1, \dots, p_j - 1$ for which $p_j \mid x + m_j d$. In our proof first we consider such cases when it is not specified which terms of the progression $x, x+d, \dots, x+(k-1)d$ are divisible by 2 and 3. Then we write $m_j = *$ for $j = 1, 2$. In such a case we say that the distribution of $p_1, \dots, p_{\pi(k-1)}$ among the a_i resp $a_i x_i^n$ corresponds to the $\pi(k-1)$-tuple $(m_{\pi(k-1)}, \dots, m_1)$. Note that in fact we shall need a kind of "negative" information: the location of the coefficients a_i without "large" prime factors will be of great importance for us. The use of our tests sieving with all $\pi(k-1)$-tuples of the form $(m_{\pi(k-1)}, \dots, m_3, *, *)$ will enable us to exclude full branches of k-tuples $(a_0, a_1, \dots, a_{k-1})$ at the same time. This makes our algorithm very efficient. Our first three tests below seem to be especially efficient, at least for the range of k under consideration.

Later we shall need to specify also those terms of $x, x+d, \dots, x+(k-1)d$ which are divisible by 2 and/or 3. For $j = 1$ and 2, let

$$m_j \in \{\times, 0, 1, \dots, k-1\} \tag{24}$$

such that, as in the case $j \geq 3$, $m_j = \times$ if $p_j \nmid \Pi_k$ and m_j is one of $0, 1, \dots, k-1$ for which $p_j \mid x + m_j d$ and

$$\mathrm{ord}_{p_j}(x + m_j d) = \max_{1 \leq \ell \leq k-1} \mathrm{ord}_{p_j}(x + m_\ell d).$$

This will enable us to calculate the exact orders of the primes $p_1 = 2$ and $p_2 = 3$ in the numbers $a_i x_i^n$. Then we shall use further tests sieving first with all possible $\pi(k-1)$-tuples of the form $(m_{\pi(k-1)},\dots,m_3,m_2,*)$, $(m_{\pi(k-1)},\dots,m_3,*,m_1)$ and thereafter with tuples $(m_{\pi(k-1)},\dots,m_3,m_2,m_1)$ with m_1,m_2 satisfying (24).

In our sieves we shall use ternary equations. We shall distinguish between (n,n,n), $(n,n,3)$ and $(n,n,2)$-sieves, according as the ternary equations involved are of signature (n,n,n), $(n,n,3)$ or $(n,n,2)$.

$(\boldsymbol{n},\boldsymbol{n},\boldsymbol{n})$-sieve I. Suppose that we are dealing with a $\pi(k-1)$-tuple $T=(m_{\pi(k-1)},\dots,m_3,*,*)$. First (by the help of T) we check whether there exists an arithmetic progression i_1,i_2,i_3 with $0\le i_1<i_2<i_3\le k-1$ such that $P(a_{i_1}a_{i_2}a_{i_3})\le 3$ and $i_1\equiv i_2\equiv i_3 \pmod 3$. If there are such indices, then by Proposition 2.4 the identity $[i_1,i_2,i_3]$ implies that $3\mid x+i_1d$ (and then consequently $3\mid x+i_2d, x+i_3d$) must be valid, otherwise we are done. Then we apply an exhaustive search for indices i_4,i_5 with which some appropriately chosen identities of the form (4) lead to a contradiction. For example, assume that $P(a_2a_5a_8)\le 3$. Then by $[2,5,8]$ we know that $3\mid x+2d, x+5d, x+8d$. Suppose further that $P(a_4a_6)\le 3$. Then $\gcd(x,d)=1$ shows that $P(a_4a_6)\le 2$. Hence, as exactly one of $\mathrm{ord}_3(x+2d)\ge 2$, $\mathrm{ord}_3(x+5d)\ge 2$, $\mathrm{ord}_3(x+8d)\ge 2$ holds, one of the identities $[2,4,5],[5,6,8],[2,6,8]$ (again by Proposition 2.4) leads to a contradiction.

After having checked all the possible $\pi(k-1)$-tuples of the form $(m_{\pi(k-1)},\dots,m_3,*,*)$ and all the possible triples (i_1,i_2,i_3) in question, we exclude those tuples T which lead in this way to a contradiction.

As an example, take $k=15$ and let

$$T=(0,3,0,\times,*,*).$$

Then we have $P(a_2a_4a_5a_6a_8)\le 3$, and by the previous argument T can be excluded.

$(\boldsymbol{n},\boldsymbol{n},\mathbf{3})$-sieve. Suppose that a $\pi(k-1)$-tuple T survives the previous test. Then we try to find an index i_0 and a difference d_0 with $P(d_0)\le 3$, $i_0-2d_0\ge 0$ and $i_0+2d_0\le k-1$ such that $P(a_{i_0-2d_0}a_{i_0-d_0}a_{i_0+d_0}a_{i_0+2d_0})\le 3$. Let $D=\gcd(x+(i_0-2d_0)d, d_0d)$. Obviously, $\gcd(x,d)=1$ and $P(d_0)\le 3$ imply that $P(D)\le 3$. Hence as $P(a_{i_0-2d_0}a_{i_0-d_0}a_{i_0+d_0}a_{i_0+2d_0})\le 3$ the equation

$$\frac{(x+(i_0-2d_0)d)}{D}\frac{(x+(i_0-d_0)d)}{D}\frac{(x+(i_0+d_0)d)}{D}\frac{(x+(i_0+2d_0)d)}{D}= \\ =\frac{a_{i_0-2d_0}a_{i_0-d_0}a_{i_0+d_0}a_{i_0+2d_0}(x_{i_0-2d_0}x_{i_0-d_0}x_{i_0+d_0}x_{i_0+2d_0})^n}{D^4}$$

yields a contradiction using Proposition 2.3. We check all the possible i_0,d_0, and exclude again all the T leading in this way to a contradiction.

To see an example, let $k = 15$ and

$$T = (0, 3, 4, 2, *, *).$$

Note that T survives the previous test. We have $P(a_5a_6a_8a_9) \leq 3$, hence we can take $i_0 = 7$ and $d_0 = 1$, and by the above test T can be excluded.

$(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n})$-sieve II. Consider a $\pi(k-1)$-tuple $T = (m_{\pi(k-1)}, \dots, m_3, *, *)$ which is not excluded by the previous tests. We let m_1 run through the set $\{\times, 0, 1, \dots, k-1\}$ and examine all $\pi(k-1)$-tuples of the form $T' = (m_{\pi(k-1)}, \dots, m_3, *, m_1)$. We perform an exhaustive search to find an identity of the form $[i_1, i_2, i_3]$ leading to a ternary equation of the shape $AX^n + BY^n = Z^n$ such that $\gcd(A, B) = 1$, and AB is either of the form 2^u3^v or 2^u5^v, with $u \geq 4$ in both cases. If we succeed, then the corresponding $\pi(k-1)$-tuple can be excluded by Proposition 2.5.

As an example, choose $k = 15$ and

$$T' = (0, 3, 1, 4, *, 11).$$

Note that this $\pi(k-1)$-tuple cannot be excluded by the previous tests. However, taking the identity $[2, 10, 11]$, after cancelling an appropriate power of 3 we get a ternary equation of the form $AX^n + BY^n = Z^n$ with $\gcd(A, B) = 1$ and $AB = 2^u3^v$, $u \geq 4$. Hence we can exclude T'.

$(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{2})$-sieve I. Suppose that a $\pi(k-1)$-tuple $T' = (m_{\pi(k-1)}, \dots, m_3, *, m_1)$ passes the previous tests. Then we consider all $\pi(k-1)$-tuples of the form $T^* = (m_{\pi(k-1)}, \dots, m_2, m_1)$ with $m_2 \in \{\times, 0, 1, \dots, k-1\}$. We search for an identity of the form $[j_2, j_3] \times [j_1, j_4]$ which leads to a ternary equation of the shape $AX^n + BY^n = CZ^2$ such that $\gcd(A, B, C) = 1$ and one of the following holds: $AB = 2^u$ $(u \neq 1)$, $C = 3^v$; $AB = 1$, $C \in \{2, 6\}$; $AB = 2^up^v$ $(u \neq 1, p \in \{11, 19\})$, $C = 1$. Then applying Proposition 2.1, the corresponding T^* $\pi(k-1)$-tuple can be excluded.

For example, choose again $k = 15$, and take

$$T^* = (0, 3, 1, 2, 0, 3).$$

Note that T^* passes all the previous sieves. However, the identity $[5, 10] \times [4, 11]$ gives rise to a ternary equation of the form $X^n + 4Y^n = 3Z^2$, which leads to a contradiction, as explained above.

$(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{2})$-sieve II. Assume that a $\pi(k-1)$-tuple T^* survives the previous tests. Then we try to find again an identity of the form $[j_2, j_3] \times [j_1, j_4]$, leading to a ternary equation $AX^n + BY^n = 2Z^2$ with $AB = 5^u$. Then Proposition 2.1 implies that $n = 7$. We collect these $\pi(k-1)$-tuples T^* to a set S, and make a note that in their cases the exponent $n = 7$ has to be handled separately.

As an example, let $k = 15$ and let

$$T^* = (0, 3, 4, 1, 8, 3).$$

As one can easily see, T^* survives the previous tests. However, after cancellations, the identity $[5, 6] \times [2, 9]$ leads to a ternary equation of the shape $X^n + 5^u Y^n = 2Z^2$ with $u > 0$, and we can put T^* into S.

$(\boldsymbol{n}, \boldsymbol{n}, \mathbf{2})$-sieve III. Assume that a $\pi(k-1)$-tuple T^* survives the previous tests. Then we search for an identity $[j_2, j_3] \times [j_1, j_4]$ such that the implied ternary equation satisfies the conditions of Proposition 2.2. Then this proposition and the subsequent Remark yield that n is (explicitly) bounded for the case corresponding to T^*. We put these $\pi(k-1)$-tuples T^* into the set S, and to each of them we attach the list of the corresponding "exceptional" exponents, to be checked later.

For example, let $k = 15$ and

$$T^* = (0, 3, 1, 4, 0, 0).$$

As one can check, this $\pi(k-1)$-tuple passes each earlier sieve. However, the identity $[6, 11] \times [3, 14]$ gives (after cancellations) a ternary equation of the shape $X^n + 5^u Y^n = Z^2$ with $11 \mid XY$, and we can put T^* into S.

After executing the above procedures, we could exclude all the $\pi(k-1)$-tuples, for all values of k, up to very few exceptions. In those cases, beside fixing the terms which are divisible by the highest powers of 2 and 3, respectively, we also fix the terms which are divisible by the highest powers of 5 and 7, respectively. Then we execute the previous tests once again. As now we have more information, because of cancellations (of 5-s and 7-s) we are able to exclude (or to put in S) these cases as well.

It remains to check the $\pi(k-1)$-tuples T^* in S for some small values of the exponent n. This can be done very easily by the following local argument.

Local sieve. For each element in S for the corresponding remaining values of n (obtained by using Proposition 2.1, Proposition 2.2 and the subsequent Remark) we consider the problem locally. For each such n we choose a prime q of the form $q = tn + 1$, with t as small as possible. For example, in the cases $n = 11, 13, 17, 19, 23$ we take $q = 23, 53, 103, 191, 47$, respectively. Then we check the putative arithmetic progressions modulo q in the following way. By the choice of the corresponding modulus, the use of Euler-Fermat theorem guarantees that x_i^n may assume only very few values modulo q. Checking all the cases one-by-one and using that the numbers $a_i x_i^n$ $(i = 0, 1, \ldots, k-1)$

should be consecutive terms of an arithmetic progression, we get a contradiction in each case.

To illustrate the local argument, chose $k = 15$, $n = 23$ and take the $\pi(k-1)$-tuple

$$(0, 3, 1, 4, 0, 0)$$

from S. Observe that the 23rd powers modulo 47 are exactly $-1, 0, 1$. Hence in this case the putative progression $a_i x_i^{23}$, $(i = 0, 1, \ldots, 14)$, should be of the form

$$\pm 2^{\alpha_0} 3^{\beta_0} 13^{\nu_0}, \pm 7^{\delta_1}, \pm 2, \pm 3 \cdot 11^{\varepsilon_3}, \pm 2^2 \cdot 5^{\gamma_4}, \pm 1, \pm 2 \cdot 3, \pm 1, \pm 2^3 7^{\delta_8}, \pm 3^2 5^{\gamma_9},$$

$$\pm 2, \pm 1, \pm 2^2 3, \pm 13^{\nu_{13}}, \pm 2 \cdot 5^{\gamma_{14}} 11^{\varepsilon_{14}}$$

modulo 47; with non-negative exponents smaller than 23 and with the possible diversion that at most one of the terms can be equal to 0. However, as one can easily check even by hand, such an arithmetic progression does not exist. In all other cases a similar argument works, and this completes the proof. $\square$

Proof of Theorem 1.3. Let (x, d, k, b, y) be a solution of (1) with $n = 5$. For $d = 1$, each factor $x + id$ in (1) must be positive or negative. Then we can reduce equation (1) to the case $x > 0$, and Proposition 2.6 applies to obtain the solutions listed in the theorem.

In what follows, we assume that $d \geq 2$. Further, if $k \leq 11$, in view of Lemma 2.1 we can restrict ourselves to the case $7 \mid a_0 \ldots a_{k-1}$.

For $8 \leq k \leq 13$, most of our work in proving Theorem 1.3 is concentrated in treating $k = 8$. We note that the above sieves can be utilized to prove our theorem for larger values of k only. For $k \leq 13$ too many exceptions would remain after using our sieves. Hence for these values of k we shall handle the arising k-tuples $(a_0, a_1, \ldots, a_{k-1})$ without using sieves, tests and computers.

The case $\boldsymbol{k = 8}$. If $7 \mid a_0, a_7$ then omitting in (1) x and $x + 7d$, we arrive at the case $k = 6$, and by Lemma 2.1 we get $x + d = -5$, $d = 2$. This yields the solution $(x, d) = (-7, 2)$. If $7 \mid a_1$ or $7 \mid a_6$, then we omit the factors $x, x + d$ resp. $x + 6d, x + 7d$ and we obtain in a similar way the solutions $(x, d) = (-9, 2), (-5, 2)$.

It remains the case $7 \mid a_2 \ldots a_5$. By symmetry it suffices to consider the case $7 \mid a_2 a_3$.

First suppose that $7 \mid a_2$. If $5 \nmid a_0 \ldots a_7$, then Lemma 2.1 applied to $\Pi(3, 4, 5, 6, 7)$ shows that there is no solution. If $5 \mid x + 2d$, then $5 \mid x + 7d$ and for $(i_1, i_2, i_3, i_4) = (3, 4, 5, 6)$ we get

$$\Pi(i_1, i_2, i_3, i_4) = b_1 y_1^5, \tag{25}$$

where b_1, y_1 are non-zero integers with $P(b_1) \leq 3$. Then Lemma 2.2 gives that either $(x+3d, d) = (-3, 2)$ which leads to the solution $(x, d) = (-9, 2)$ or, up to symmetry,

$$(a_3, a_4, a_5, a_6) = (4, 3, 2, 1) \quad \text{or} \quad (9, 4, 1, 6).$$

If (a_3, a_4, a_5, a_6) equals $(4, 3, 2, 1)$ or $(1, 2, 3, 4)$, then applying Proposition 2.7 to $[0, 3, 6]$ resp. to $[1, 2, 3]$, we arrive at a contradiction. In the remaining cases Proposition 2.7 can be applied to $[1, 3, 5]$ or $[0, 1, 3]$ and we get again a contradiction.

Next assume that $5 \mid x$. If $3 \nmid \Pi_8$ or $3 \mid x$, we can apply Proposition 2.7 to $[1, 4, 7]$. Otherwise, Proposition 2.7 can be applied to $[1, 4, 7]$, $[4, 6, 7]$ or $[1, 3, 4]$ to obtain a contradiction.

Let $5 \mid x + d$. If $3 \nmid \Pi_8$ or $3 \mid x$, one of the equations $[3, 4, 5]$, $[0, 3, 4]$, $[5, 6, 7]$, $[4, 5, 7]$ leads to a contradiction by Proposition 2.7. In the remaining cases at least one of the equations $[3, 4, 5]$, $[0, 1, 3]$, $[4, 5, 7]$, $[0, 3, 6]$, $[3, 5, 7]$, $[0, 2, 4]$ is not solvable by Proposition 2.7.

Let now $5 \mid x + 3d$. If $3 \nmid \Pi_8$ or $3 \mid x(x + 2d)$, then using Proposition 2.7, equation $[1, 4, 7]$ leads to a contradiction. If $3 \mid x + d$, we get the equation (25) with $(i_1, i_2, i_3, i_4) = (4, 5, 6, 7)$. Then Lemma 2.2 gives that either $(x + 4d, d) = (-3, 2)$ which does not yield any solution of (1) or, up to symmetry,

$$(a_4, a_5, a_6, a_7) = (4, 3, 2, 1) \quad \text{or} \quad (9, 4, 1, 6).$$

It is easy to verify that only the second option can occur. Then $[0, 3, 6]$ or $[1, 4, 5]$ has no solution, according as (a_4, a_5, a_6, a_7) equals $(9, 4, 1, 6)$ resp. $(6, 1, 4, 9)$.

Finally assume that $5 \mid x+4d$. Then applying Lemma 2.2 to equation (25) with $(i_1, i_2, i_3, i_4) = (1, 3, 5, 7)$ we get that either $(x+d, d) = (-3, 2)$ which yields the solution $(x, d) = (-5, 2)$ of (1) or, up to symmetry,

$$(a_1, a_3, a_5, a_7) = (4, 3, 2, 1) \quad \text{or} \quad (9, 4, 1, 6).$$

It follows that in each case $x + d, x + 3d, x + 5d$ and $x + 7d$ are all divisible by 4 which contradicts the assumption that $\gcd(x, d) = 1$.

Next consider the case $7 \mid x + 3d$. If $5 \nmid a_0 \dots a_7$ or if $5 \mid x + 3d$ then we have (25) with $(i_1, i_2, i_3, i_4) = (4, 5, 6, 7)$. Then, by Lemma 2.2, (a_4, a_5, a_6, a_7) equals $(4, 3, 2, 1)$, $(1, 2, 3, 4)$, $(9, 4, 1, 6)$ or $(6, 1, 4, 9)$. Now Proposition 2.7 proves that $[1, 4, 7]$, $[2, 3, 4]$, $[0, 1, 2]$ resp. $[1, 4, 5]$ is not solvable.

Let now $5 \mid x$. If $3 \nmid x + d$ then Proposition 2.7 applies to $[1, 4, 7]$, leading to a contradiction. If $3 \mid x + d$, then by Proposition 2.7 at least one of the equations $[2, 4, 6]$, $[1, 4, 7]$, $[4, 6, 7]$, $[1, 2, 4]$ has no solution.

Assume now that $5 \mid x+d$. If $3 \nmid \Pi_8$ or $3 \mid x$, then by Proposition 2.7, one of the equations $[0,2,4]$, $[2,3,4]$ and $[5,6,7]$ has no solution satisfying (1). Let now $3 \mid x+d$. If x is odd then equation $[0,1,2]$ is not solvable by Proposition 2.7. Otherwise, if x is even then by $\gcd(x,d)=1$ d is odd, whence $2^2 \mid x$ or $2^2 \mid x+2d$. If $3^2 \nmid x+7d$ or $3^2 \mid x+7d$ and $2^2 \mid x$ then Proposition 2.7 shows that $[4,5,7]$ resp. $[2,4,5]$ is not solvable. When $3^2 \mid x+7d$ and $2^2 \mid x+2d$, then using the fact that

$$X^5 \equiv 0, \pm 1 \pmod{11} \tag{26}$$

for any integer X, we deduce that $[1,4,5]$ is not solvable $\pmod{11}$.

Next let $3 \mid x+2d$. If x is odd or $\mathrm{ord}_2(x)=\mathrm{ord}_2(x+4d)$, then in view of Proposition 2.7 $[0,2,4]$ has no solution. As $\gcd(x,d)=1$, it remains the case when $2^3 \mid x$ or $2^3 \mid x+4d$. If $3^2 \nmid x+2d$ and $3^2 \nmid x+5d$, then $[2,4,5]$ is not solvable by Proposition 2.7.

Assume that $3^2 \mid x+2d$. If $2^3 \mid x$, then $[4,5,7]$ yields the only solution

$$x_4 \equiv x_5 \equiv x_7 \equiv \pm 1 \pmod{11}.$$

Together with (3) this gives $d=1$ which is excluded. If $2^3 \mid x+4d$, then $[0,5,7]$ is not solvable $\pmod{11}$. Finally, consider the case $3^2 \mid x+5d$. If $2^3 \mid x+4d$, then Proposition 2.7 shows that equation $[1,4,7]$ is not solvable. By assumption we have $5 \mid x+6d$. If $2^3 \mid x$, then $[1,4,7]$ or $[2,4,6]$ is not solvable $\pmod{11}$, according as $5^2 \mid x+6d$ or not.

Let now $5 \mid x+2d$. If $3 \nmid \Pi_8$, then solving $[4,5,6]$ by means of Proposition 2.7 we do not get any solution for (1). First assume that $3 \mid x+d$. Then, by Proposition 2.7, $[0,3,6]$ or $[4,5,6]$ has no solution, according as $2^2 \nmid x$ or $2^2 \mid x$. Next let $3 \mid x+2d$. Then Proposition 2.7 implies that $[0,1,4]$, $[0,4,6]$ or $[1,2,4]$ is not solvable, according as $2^3 \mid x$, $2^3 \mid x+4d$ or $2^3 \nmid x$ and $2^3 \nmid x+4d$. Assume now that $3 \mid x$. If $2^3 \nmid x+d$ and $2^3 \nmid x+5d$ then $[1,3,5]$ is not solvable in view of Proposition 2.7. It remains the case $2^3 \mid x+d$ or $2^3 \mid x+5d$. Then Proposition 2.7 implies that $[0,3,6]$, $[3,4,6]$ or $[1,4,5]$ is not solvable, according as $\mathrm{ord}_3(x)=\mathrm{ord}_3(x+6d)=1$, $3^2 \mid x$ or $3^2 \mid x+6d$ and $2^3 \mid x+5d$. If $3^2 \mid x+6d$ and $2^3 \mid x+d$, then Proposition 2.5 proves that $[0,1,4]$ has no solution.

Finally, assume that $5 \mid x+4d$. If $3 \nmid \Pi_8$ or $3 \mid x+2d$, then at least one of the equations $[0,3,6]$, $[1,4,7]$ is not solvable by Proposition 2.7. If $3 \mid x$, then, by Proposition 2.7, $[1,2,5]$, $[1,5,7]$ or $[1,3,5]$ is not solvable, according as $2^3 \mid x+d$, $2^3 \mid x+5d$ or $2^3 \nmid x+d$ and $2^3 \nmid x+5d$. If $3 \mid x+d$, then $[0,2,6]$, $[2,5,6]$ or $[2,4,6]$ has no solution, according as $2^3 \mid x+2d$, $2^3 \mid x+6d$ or $2^3 \nmid x+2d$ and $2^3 \nmid x+6d$. This completes the proof of the case $k=8$.

The cases $k = 9, 10, 11$. In view of $P(b) \leq 7$, (1) implies (3) with $P(a_i) \leq 7$ for each i. Hence we deduce from (1) that

$$\Pi(0, 1, \ldots, k-2) = b_2 y_2^5 \tag{27}$$

where b_2, y_2 are non-zero integers with $P(b_2) \leq 7$. We can now proceed by induction on k. For $k = 9$, we apply to (27) our results proved above in the case $k = 8$ and we infer that all the solutions of (1) with $d \geq 2$ are given by $d = 2$, $x \in \{-9, -7\}$. For $k = 10$, we obtain similarly that $d = 2$, $x = -9$, while, for $k = 11$, we do not get any solution for (1).

The cases $k = 12, 13$. First suppose that at most one factor, say $x + id$, is divisible by 11. Then $11 \nmid a_i$, and we get (27). Using again induction on k, we infer that in these cases (1) has no solution. If two factors, say $x + id$ and $x + jd$ with $i < j$, are divisible by 11, then we deduce from (1) that

$$\Pi(i+1, \ldots, j-1) = b_3 y_3^5, \tag{28}$$

where $j = i + 11$ and b_3, y_3 are non-zero integers with $P(b_3) \leq 7$. We can now apply our results obtained for $k = 10$ and it follows that no new solutions of (1) arise.

The cases $k \geq 14$. From this point on it is definitely worth algorithmizing the proof and using a computer. We execute the following tests. As they are rather similar to those used in case of $n \geq 7$, we apply the same notation.

$(5, 5, 5)$-sieve I-II. We apply the sieves **(n, n, n)-sieve I** and **(n, n, n)-sieve II** like in case of $n \geq 7$, but consecutively. As the underlying Propositions 2.4 and 2.5 are valid also for $n = 5$, this can be done without any restrictions.

$(5, 5, 5)$-sieve III. From this point on we work with the set T^* (see the corresponding part of the proof of Theorem 1.2). For each $\pi(k-1)$-tuple in T^* we check whether it is possible to find three terms of the arithmetic progressions under consideration, such that their corresponding linear combination leads to an equation of the form

$$X^n + Y^n = CZ^n$$

with $P(C) \leq 5$. If we can find such terms, then the corresponding $\pi(k-1)$-tuple T^* can be excluded by Proposition 2.7. (We can easily take care of the cases corresponding to part (i) of the proposition.) If a $\pi(k-1)$-tuple T^* cannot be excluded, we put it into a set S.

Sieve modulo 11. Similarly as in **Local sieve**, we test all elements of S locally. In this case we can obviously use the prime 11. By the help of the same method as in the proof of Theorem 1.2, all $\pi(k-1)$-tuples in S can be excluded, and the proof is complete. □

Proof of Theorem 1.1. We must prove that for $3 < k < 35$ and $b = 1$, equation (1) has no solution in positive integers x, d, y and n. Suppose that such a solution exists. By the result of Erdős and Selfridge we may assume that $d > 1$. Further, as was mentioned earlier, without loss of generality we may assume that n is prime. If $n = 2$ or $n = 3$, then the statement immediately follows from Theorem B and Theorem C, respectively. In case of $n = 5$, Theorem 1.1 is a consequence of Theorem A and Theorem 1.3. Finally, for any prime $n \geq 7$ Theorem A together with Theorem 1.2 imply the assertion. □

4. Acknowledgements

The authors wish to thank Professor M. A. Bennett for the stimulating correspondence and for making a few corrections of grammatical nature in the manuscript.

References

[1] M. A. Bennett, Recipes for ternary Diophantine equations of signature (p, p, k), *Proc. RIMS Kokyuroku (Kyoto)* **1319** (2003), 51–55.

[2] M. A Bennett, Powers from five terms in arithmetic progression, in: Diophantine Equations, Narosa Publ. House, New Delhi, 2008. pp. 53–57.

[3] M. A Bennett, N. Bruin, K. Győry and L. Hajdu, Powers from products of consecutive terms in arithmetic progression, *Proc. London Math. Soc.* **92** (2006), 273–306.

[4] M. A Bennett, K. Győry and Á. Pintér, On the diophantine equation $1^k + 2^k + \ldots + x^k = y^n$, *Compositio Math.* **140** (2004), 1417–1431.

[5] M. A. Bennett and C. Skinner, Ternary Diophantine equations via Galois representations and modular forms, *Canad. J. Math.* **56** (2004) 23-54.

[6] M. A. Bennett, V. Vatsal and S. Yazdani, Ternary Diophantine equations of signature $(p, p, 3)$, *Compositio Math.* **140** (2004), 1399–1416.

[7] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[8] H. Cohen, Number theory. Vol. II. Analytic and modern tools, *Graduate Texts in Mathematics*, **2**40, Springer, New York, 2007. xxiv+596 pp.

[9] H. Darmon and L. Merel, Winding quotients and some variants of Fermat's Last Theorem, *J. Reine Angew. Math.* **490** (1997), 81–100.

[10] P. Dénes, Über die diophantische Gleichung $x^l + y^l = cz^l$, *Acta Math.* **88** (1952), 241–251.

[11] L. E. Dickson, History of the theory of numbers. Vol. II: Diophantine analysis, Chelsea Publishing Co., New York 1966.

[12] P. Erdős and J. L. Selfridge, The product of consecutive integers is never a power, *Illinois J. Math.* **19** (1975), 292-301.

[13] K. Győry, On the diophantine equation $n(n+1)\dots(n+k-1) = bx^l$, *Acta Arith.* **83** (1998), 87–92.

[14] K. Győry, Power values of products of consecutive integers and binomial coefficients, *Number Theory and Its Applications*, Kluwer Acad. Publ. 1999, 145–156.

[15] K. Győry, L. Hajdu and N. Saradha, On the Diophantine equation $n(n+d)\dots(n+(k-1)d) = by^l$, *Canad. Math. Bull.* **47** (2004), 373-388. Correction: *Canad. Math. Bull.* **48** (2005), 636.

[16] L. Hajdu, Sz. Tengely and R. Tijdeman, Cubes in products of terms in arithmetic progression (to appear).

[17] N. Hirata-Kohno, S. Laishram, T. Shorey and R. Tijdeman, An extension of a theorem of Euler, *Acta Arith.* **129** (2007), 71–102.

[18] A. Kraus, Majorations effectives pour léquation de Fermat généralisée, *Canad. J. Math.* **49** (1997), 1139-1161.

[19] A. Kraus, On the equation $x^p + y^q = z^r$: a survey, *Ramanujan J.* **3** (1999), 315-333.

[20] R. Obláth, Über das Produkt fünf aufeinander folgender Zahlen in einer arithmetischen Reiche, *Publ. Math. Debrecen* **1** (1950), 222-226.

[21] R. Obláth, Eine Bemerkung über Produkte aufeinander folgender Zahlen, *J. Indian Math. Soc.* **15** (1951), 135-139.

[22] K. Ribet, On the equation $a^p + 2^\alpha b^p + c^p = 0$, *Acta Arith.* **79** (1997), 7–16.

[23] J. W. Sander, Rational points on a class of superelliptic curves, *J. London Math. Soc.* **59** (1999), 422-434.

[24] N. Saradha, On perfect powers in products with terms from arithmetic progressions, *Acta Arith.* **82** (1997), 147–172.

[25] N. Saradha and T.N. Shorey, Almost perfect powers in arithmetic progression, *Acta Arith.* **99** (2001), 363–388.

[26] T. N. Shorey, Powers in arithmetic progression, in: A Panorama in Number Theory (G. Wüstholz, ed.), Cambridge University Press, Cambridge, 2002, 325-336.

[27] T. N. Shorey, Powers in arithmetic progression (II), in: New Aspects of Analytic Number Theory, Kyoto 2002, 202-214.

[28] Sz. Tengely, Note on the paper "An extension of a theorem of Euler" by Hirata-Kohno et al. (submitted).

[29] R. Tijdeman, Diophantine equations and diophantine approximations, in: Number Theory and Applications, Kluwer Acad. Press, 1989, 215-243.

[30] A. Wiles, Modular elliptic curves and Fermat's Last Theorem, *Ann. Math.* **141** (1995), 443–551.

K. Győry, L. Hajdu, Á. Pintér
University of Debrecen, Institute of Mathematics
and the Number Theory Research Group
of the Hungarian Academy of Sciences
Debrecen
P.O. Box 12.
H-4010
Hungary
E-mail address: {gyory|hajdul|apinter}@math.klte.hu