Metrical neighborhood sequences in $\mathbb{Z}^n$

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Abstract

Digital metrics on the digital space play an important role in several branches of discrete mathematics, e.g. in discrete geometry or digital image processing. We perform an overall analysis on some properties of neighborhood sequences which induce metrics on $\mathbb{Z}^n$.

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1 Introduction

Motions on the digital space play an important role in several parts of discrete mathematics, including discrete geometry and digital image processing. The most important motions in $\mathbb{Z}^2$ are based upon the classical 4-neighborhood and 8-neighborhood relations. These relations lead to the so called cityblock (or von Neumann) and the chessboard (or Moore) motions, respectively. The alternate use of these neighborhood relations gives rise to the octagonal distance. These motions and the induced distance functions were systematically investigated in the classical paper of Rosenfeld and Pfaltz (1968). By allowing any periodic mixture of the 4- and 8 neighborhood relations, Das et al. (1987a) introduced the concept of periodic neighborhood sequences. They also extended this notion to $\mathbb{Z}^n$. Several papers are devoted to the description of the properties of such sequences, see e.g. (Das, 1990; Das and Chatterji, 1990; Das et al., 1987b) and the references given there. Later, Fazekas et al. (2002) extended the theory to the general case, i.e. when any (not necessary periodic) sequences are considered. The use of such sequences provide a more flexible tool than the previous ones. For example, A. Hajdu and L. Hajdu (2004) could obtain digital metrics on $\mathbb{Z}^2$ based upon such sequences, which yield the best approximation to the Euclidean distance in some sense. Using periodic sequences, only some parts of such sequences can be given, see e.g. (Das, 1992; Mukherjee et al., 2000).

Those neighborhood sequences which generate metrics on the digital space $\mathbb{Z}^n$ naturally play a special role in the above mentioned problems and areas. In this paper we perform an overall analysis on the structural and individual properties of these sequences. It turns out that in 2D the set of such sequences has a nice algebraic structure under a natural partial ordering relation (Section 3), and that in any dimension it has some interesting topological properties, as well (Section 4). We also prove that if a neighborhood sequence A generates a metric, then each symbol in A has a density (Section 5). Finally, we give some data about the prefixes of metrical neighborhood sequences in $\mathbb{Z}^n$ (Section 6).

2 Basic concepts and notation

In this section we introduce some standard notation concerning neighborhood sequences, (see e.g. (Das et al., 1987a; Fazekas et al., 2002)).

For the whole paper let $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the set of integers and positive integers, respectively.

Let $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$ with $0 \le m \le n$. The points $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ in $\mathbb{Z}^n$ are m-neighbors, if the following two conditions hold:

- $|p_i - q_i| \le 1 \quad (1 \le i \le n)$,
- $\sum\limits_{i=1}^{n} |p_i - q_i| \le m$.

The sequence $A = (A(i))_{i=1}^{\infty}$, where $A(i) \in \{1, \ldots, n\}$ for all $i \in \mathbb{Z}^+$, is called an n-dimensional (shortly nD) neighborhood sequence. If for some non-negative integer k and $l \in \mathbb{Z}^+$ we have $A(i + l) = A(i)$ whenever $i > k$ then we briefly write

$$A = A(1)A(2)\ldots A(k)\overline{A(k+1)A(k+2)\ldots A(k+l)}.$$

In case of $k = 0$, i.e. when $A = \overline{A(1)A(2)\ldots A(l)}$, A is called periodic with period l. The set of the nD-neighborhood sequences will be denoted by S_n, while the set of periodic ones by P_n.

Let $p,\ q \in \mathbb{Z}^n$ and $A \in S_n$. The point sequence $p = p_0, p_1, \ldots, p_t = q$, where p_{i-1} and p_i are $A(i)$-neighbors in $\mathbb{Z}^n$ $(1 \leq i \leq t)$, is called an A-path from p to q of length t. The A-distance $d(p, q; A)$ of p and q is defined as the length of the shortest A-path(s) between them. As a brief notation, we also use $d(A)$ for the A-distance.

It is not true that $d(A)$ is a metric on $\mathbb{Z}^n$ for every $A \in S_n$. With the following result of Nagy (2003) we can decide whether the distance function related to A is a metric on the n-dimensional digital space, or not.

Theorem 1 (see (Nagy, 2003)) *Let $A \in S_n$, and for every $i \in \mathbb{Z}^+$ and $j \in \{1, \ldots, n\}$ put $A^{(j)}(i) = \min(A(i), j)$. Then $d(A)$ is a metric if and only if*

$$\sum_{i=1}^{k} A^{(j)}(i) \leq \sum_{i=t}^{k+t-1} A^{(j)}(i)$$

for any $k, t \in \mathbb{Z}^+$ and $j \in \{1, \ldots, n\}$.

For each integer n with $n \geq 2$ let M_n denote the set of those nD-neighborhood sequences which generate metrics on $\mathbb{Z}^n$. If $A \in M_n$ then A is called metrical.

In our structural investigations we examine the set of metrical neighborhood sequences with respect to two partial orderings, $\sqsupseteq^*$ and $\sqsupseteq$. These orderings are defined in the following way. For $A, B \in S_n$ write

$$A \sqsupseteq^* B \iff d(p, q; A) \leq d(p, q; B) \text{ for every } p, q \in \mathbb{Z}^n,$$

and set

$$A \sqsupseteq B \iff A(i) \geq B(i) \text{ for every } i \in \mathbb{Z}^+.$$

The ordering $\sqsupseteq^*$ was introduced in (Das et al., 1987a) for P_n and was investigated in (Das, 1990) and (Fazekas, 1999) later on. In (Fazekas et al., 2002) the authors extended this ordering to S_n and introduced $\sqsupseteq$, as well. From (Fazekas et al., 2002) we know that

$$A \sqsupseteq^* B \iff \sum_{i=1}^{k} A^{(j)}(i) \geq \sum_{i=1}^{k} B^{(j)}(i) \text{ for any } k \in \mathbb{Z}^+ \text{ and } j \in \{1, \ldots, n\},$$

and that the ordering $\sqsupseteq$ is a proper refinement of $\sqsupseteq^*$.

Now we recall a few basic concepts and facts from lattice theory. They will be used throughout the paper without any further reference. Let H be a partially ordered set. We say that H is a lattice, if for any $A, B \in H$ the greatest lower bound $A \wedge B$ and the least upper bound $A \vee B$ of these elements exist. If for any $S \subseteq H$ the greatest lower bound $\bigwedge S$ and the least upper bound $\bigvee S$ of S also exist, then the lattice H is called complete. It is well-known that if $\bigwedge S$ exists for all subset S of H, then $\bigvee S$ also exists for any subset, and vice versa.

The lattice H is distributive, if for any $A, B, C \in H$ we have

$$(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C) \quad \text{and} \quad (A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C).$$

As in our investigations we consider greatest lower bounds and least upper bounds both in M_n and in S_n, we use the following convention. The simple notation $\wedge$ and $\vee$ will always refer to the corresponding elements in M_n (with respect to the given ordering), and we will write $\wedge_{S_n}$ and $\vee_{S_n}$ if we work in S_n.

3 Lattices of metrical neighborhood sequences

In this section we investigate the structural behavior of the set of metrical neighborhood sequences with respect to both $\sqsupseteq^*$ and $\sqsupseteq$. We start with some basic results. First we formulate a result from (Fazekas et al., 2002) which will be a useful tool.

Lemma 2 *$(S_2, \sqsupseteq^*)$ is a complete distributive lattice. Moreover, if S is any subset of S_2 then for the sequences $A = \bigwedge_{S_2} S$ and $B = \bigvee_{S_2} S$, for any $k \in \mathbb{Z}^+$ we have*

$$\sum_{i=1}^{k} A(i) = \min \left\{ \sum_{i=1}^{k} C(i) \,\middle|\, C \in S \right\} \quad \textit{and} \quad \sum_{i=1}^{k} B(i) = \max \left\{ \sum_{i=1}^{k} C(i) \,\middle|\, C \in S \right\}.$$

PROOF. The statement is a reformulation of Theorem 3.5 from (Fazekas et al., 2002); see also its proof. □

Remark 3 *By Proposition 3.14 of (Fazekas et al., 2002) we also have that $(S_n, \sqsupseteq)$ is a complete distributive lattice for any $n \geq 2$.*

The next result shows that it is not true that any two metrical neighborhood sequences can be compared using these orderings.

Proposition 4 *The partial orderings $\sqsupseteq^*$ and $\sqsupseteq$ are not total orders on M_n.*

PROOF. Let $A = \overline{12}$, $B = \overline{11222}$. By Theorem 1 we can see that $A, B \in M_n$. Moreover, it is easy to check that A and B cannot be compared neither with $\sqsupseteq^*$, nor with $\sqsupseteq$. □

3.1 *The structure of M_n with respect to $\sqsupseteq$*

In (Fazekas et al., 2002) the authors introduced $\sqsupseteq$ to obtain better structural results for S_n and P_n than with $\sqsupseteq^*$. The following result shows the slightly surprising fact that M_n does not form a nice structure under $\sqsupseteq$.

Proposition 5 *$(M_n, \sqsupseteq)$ is not a lattice for $n \geq 2$.*

PROOF. Let $A = 1222221\overline{2}, B = 1222122\overline{2}$. By Theorem 1 we have that $A, B \in M_n$. We show that $A \wedge B$ does not exist.

Let $C = 1212121\overline{2}, D = 1122121\overline{2}$. Clearly, $C, D \in M_n$, $A \sqsupseteq C$, $B \sqsupseteq C$, $A \sqsupseteq D$, and $B \sqsupseteq D$. Moreover, neither C nor D can be the greatest lower bound of A and B in M_n, since C and D cannot be compared. Looking at the first few elements of A, B, C and D we obtain that if $A \wedge B$ exists, then we must have $A \wedge B = 1222121\ldots$. However, such a sequence cannot belong to M_n, which yields that $A \wedge B$ does not exist. $\square$

3.2 *The structure of M_n with respect to $\sqsupseteq^*$*

The situation for $(M_n, \sqsupseteq^*)$ is similar to $(M_n, \sqsupseteq)$ at least when $n \geq 3$. However, this is not that surprising, since it was shown in (Fazekas et al., 2002) that $(S_n, \sqsupseteq^*)$ is also not a lattice in this case.

Proposition 6 *$(M_n, \sqsupseteq^*)$ is not a lattice for $n \geq 3$.*

PROOF. Let n be an integer with $n \geq 3$ and put $A = \overline{13}$, $B = \overline{123}$, $C = 13\overline{2}$ and $D = 1331\overline{3}$. By Theorem 1 it is easy to check that $A, B, C \in M_n$. We also have that $A \sqsubseteq^* C$, $B \sqsubseteq^* C$, $A \sqsubseteq^* D$, $B \sqsubseteq^* D$ and $C \not\sqsubseteq^* D$, $D \not\sqsubseteq^* C$.

To prove the statement we will show that the least upper bound of A and B does not exist in M_n. Assume to the contrary that $E = A \vee B$ exists. By the existence of D, $E \neq C$. As $E \sqsubseteq^* C$ must be valid, we have $E(t) < C(t)$ for some $t \in \mathbb{Z}^+$. Without loss of generality we may assume that t is minimal with this property. A simple calculation shows that the first three elements of E has to be given by $1, 3, 2$. This yields that $t \geq 4$ and $E(t-1) = 2$, $E(t) = 1$. However, then we have $E(t-1) + E(t) < E(1) + E(2)$ which contradicts the metricity of E. Thus $A \vee B$ does not exist, and the proof is complete. $\square$

The following theorem shows that contrary to the higher dimensional case, metrical 2D-neighborhood sequences form a nice structure with respect to $\sqsupseteq^*$.

Theorem 7 *$(M_2, \sqsupseteq^*)$ is a complete lattice. Moreover, for any subset M of M_2 we have $\bigwedge M = \bigwedge_{S_2} M$.*

PROOF. Let M be an arbitrary subset of M_2. In view of Lemma 2, $\bigwedge_{S_2} M$ exists, so we put $D = (D(i))_{i=1}^{\infty} = \bigwedge_{S_2} M$. We prove that $D = \bigwedge M$ also holds, i.e. $D \in M_2$. Suppose to the contrary that $D \notin M_2$. Then by Theorem 1 there exist $k, l \in \mathbb{Z}^+$ such that

$$\sum_{i=1}^{k} D(i) > \sum_{i=l+1}^{l+k} D(i)$$

holds. Further, using Lemma 2 we get that for some $A \in M$

$$\sum_{i=1}^{l+k} D(i) = \sum_{i=1}^{l+k} A(i) \quad \text{and also} \quad \sum_{i=1}^{l} D(i) \leq \sum_{i=1}^{l} A(i).$$

From these assertions we deduce that

$$\sum_{i=1}^{k} D(i) + \sum_{i=1}^{l} D(i) > \sum_{i=1}^{l+k} D(i) = \sum_{i=1}^{l+k} A(i) \geq \sum_{i=1}^{l} D(i) + \sum_{i=l+1}^{l+k} A(i),$$

which by the metricity of A gives

$$\sum_{i=1}^{k} D(i) > \sum_{i=1}^{k} A(i).$$

However, this contradicts $D \sqsubseteq^* A$, and the theorem follows. $\square$

It is an interesting property of M_2 that while for any $A, B \in M_2$ we have $A \wedge_{S_2} B \in M_2$, the same statement does not hold for $A \vee_{S_2} B$. For example, if we choose $A = \overline{112}$, $B = \overline{111222}$ then it is easy to verify that $A, B \in M_2$ and $A \vee_{S_2} B = 112122111\ldots$, which sequence does not belong to M_2. On the other hand, the least upper bound of A and B also exists in M_2, since M_2 is a complete lattice. By Lemma 2 it is easy to determine $A \wedge B$ for any $A, B \in M_2$, but how to determine $A \vee B$? The following theorem gives an answer to this problem in a more general form.

Theorem 8 *For any $A = (A(i))_{i=1}^{\infty} \in S_2$ there exists a $B = (B(i))_{i=1}^{\infty} \in M_2$ with $B \sqsupseteq^* A$, such that for any $C \in M_2$ with $C \sqsupseteq^* A$, $C \sqsupseteq^* B$ holds.*

Moreover, $B(1) = A(1)$ *and if the first* k *elements of* B *are already given, then*

$$B(k+1) = \begin{cases} 1, & \begin{array}{l} \text{if } \sum_{i=1}^{k+1} A(i) \le \sum_{i=1}^{k} B(i) + 1 \text{ and} \\ \sum_{i=1}^{l} B(i) \le \sum_{i=k-l+2}^{k} B(i) + 1 \text{ for every } l = 1, \dots, k, \end{array} \\ 2, & \text{otherwise.} \end{cases}$$

PROOF. We show that the sequence $B = (B(i))_{i=1}^{\infty}$ defined by the inductive procedure in the statement meets the requirements of the theorem. We clearly have that

$$\sum_{i=1}^{l} B(i) \le \sum_{i=k-l+2}^{k+1} B(i) \quad \text{for any } k \in \mathbb{Z}^+ \text{ and } l \in \{1, \dots, k\},$$

whence $B \in M_2$. Further, as $\sum_{i=1}^{k} A(i) \le \sum_{i=1}^{k} B(i)$ holds for any $k \in \mathbb{Z}^+$, we also have $B \sqsupseteq^* A$.

Finally, assume that there exists a $C = (C(i))_{i=1}^{\infty} \in M_2$, such that $C \sqsupseteq^* A$, and $C \not\sqsupseteq^* B$. Then choose the minimal $t \in \mathbb{Z}^+$ for which $\sum_{i=1}^{t} B(i) > \sum_{i=1}^{t} C(i)$. We have that $t \ge 2$, $B(t) = 2$, $C(t) = 1$ and

$$\sum_{i=1}^{t-1} B(i) = \sum_{i=1}^{t-1} C(i). \tag{1}$$

Since $B(t) = 2$, from the inductive condition we infer that either $\sum_{i=1}^{t} A(i) > \sum_{i=1}^{t-1} B(i) + 1$, or $\sum_{i=1}^{l} B(i) > \sum_{i=t-l+1}^{t-1} B(i) + 1$ for some $l \in \{1, \dots, t-1\}$. In the first case (1) and $C(t) = 1$ yield that $C \not\sqsupseteq^* A$, which is a contradiction. In the second case, using the appropriate l, by the minimality of t

$$\sum_{i=1}^{t-l} B(i) \le \sum_{i=1}^{t-l} C(i) \tag{2}$$

holds. We also have

$$\sum_{i=1}^{l} C(i) \ge \sum_{i=1}^{l} B(i) > \sum_{i=t-l+1}^{t-1} B(i) + 1. \tag{3}$$

Putting together (1) and (2) we obtain

$$\sum_{i=t-l+1}^{t-1} B(i) \ge \sum_{i=t-l+1}^{t-1} C(i). \tag{4}$$

Combining (3) and (4) we get

$$\sum_{i=1}^{l} C(i) > \sum_{i=t-l+1}^{t-1} C(i) + 1 = \sum_{i=t-l+1}^{t} C(i),$$

which contradicts $C \in M_2$, and the theorem follows. $\square$

Let us define the metrical closure of the neighborhood sequence $A \in S_2$ as the sequence B given by the above theorem. Then in case of $B_1, B_2 \in M_2$, $B_1 \vee B_2$ is clearly the metrical closure of $B_1 \vee_{S_2} B_2$. Now we provide an infinite procedure which produces the metrical closure of $A = (A(i))_{i=1}^{\infty} \in S_2$. To simplify the description, we define the concept of switching and switching back as changing a sequence element from 1 to 2 and vice versa, respectively. Moreover, we call a finite word $(C(i))_{i=1}^{k}$ metrical if $\sum_{i=1}^{l} C(i) \le \sum_{i=k-l+1}^{k} C(i)$ holds for every $l \in \{1, \dots, k\}$.

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1: c ← 0 {Invoking the counting variable for switching.}
2: k ← 1 {Invoking the slice length for checking metricity.}
3: B(k) ← A(k) {Setting the next element of the metrical closure B of A.}
4: if (B(i))_{i=1}^k is not metrical then {Checking metricity for k.}
5:    B(k) ← 2 {Making (B(i))_{i=1}^k metrical by switching.}
6:    c ← c + 1 {Updating the number of switchings.}
7: else if B(k) = 2 and c > 0 then {Switching back if possible.}
8:    if '(B(i))_{i=1}^{k-1} 1' is metrical then {Preserving metricity.}
9:       B(k) ← 1 {Switching back B(k).}
10:      c ← c − 1 {Updating the number of switchings.}
11:   end if
12: end if
13: k ← k + 1 {Increasing the slice length for the next metricity check.}
14: go to 3: {Finding the next element of B.}
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Note that in the above algorithm the counter of switchings for the k-th step can be calculated as $c = \#\{l \mid B(l) = 2,\ 1 \le l \le k\} - \#\{l \mid A(l) = 2,\ 1 \le l \le k\}$. As one can see, this procedure is a kind of greedy algorithm: it keeps c as small as possible, beside keeping the metricity. Using Theorem 8, it is easy to check that this algorithm is correct. However, for the convenience of the reader we include a simple example to illustrate how the algorithm works.

Example 9 *Let $A = 12111122211\overline{2} \in S_2$. In Table 1 we can follow the steps of the algorithm for creating the metrical closure of the non-metrical sequence A. It can be observed how the counter c for the number of switchings changes and how the metrical behavior of $B(k)$ is guaranteed by the algorithm. Especially, for $k = 4, 6, 10$ we can see how metricity is achieved by choosing the 2 value at these indices, while for $k = 7, 9, 13$ we can see examples for switching back* *to obtain the least upper bound. For $k = 8$ we can see the case when switching back is not possible without violating metricity.*

Table 1
Generating algorithmically the metrical closure of the non-metrical neighborhood sequence $A = 12111122211\overline{2}$. Parameters k and c denote the number of steps and switchings, respectively.

	k														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$k \geq 15$
$A(k)$	1	2	1	1	1	1	2	2	2	1	1	2	2	2	2
$B(k)$	1	2	1	2	1	2	1	2	1	2	1	2	1	2	2
c	0	0	0	1	1	2	1	1	0	1	1	1	0	0	0

By the help of Theorem 8 and the above algorithm, we can easily show that the distributive property does not hold for the lattice $(M_2, \sqsupseteq^*)$.

Proposition 10 *The lattice $(M_2, \sqsupseteq^*)$ is not distributive.*

PROOF. Let $A = 11211211\overline{2}$, $B = 111222111\overline{2}$ and $C = 112211221\overline{2}$. By Theorem 1, Lemma 2 and Theorem 8 we obtain that $A, B, C \in M_2$, $A \vee_{S_2} B = 112122111\overline{2}$, and $A \vee B = 1121221121\overline{2}$. Moreover, we get $(A \vee B) \wedge C = 1121212121\overline{2}$. On the other hand, $(A \wedge C) \vee (B \wedge C) = 112121211\overline{2}$. That is, $(A \vee B) \wedge C \neq (A \wedge C) \vee (B \wedge C)$ in M_2, whence the distributive property fails. □

To close our investigations on the lattice structure of M_2 we present Figure 1 to illustrate how the lattice $(M_2, \sqsupseteq^*)$ is situated in $(S_2, \sqsupseteq^*)$.

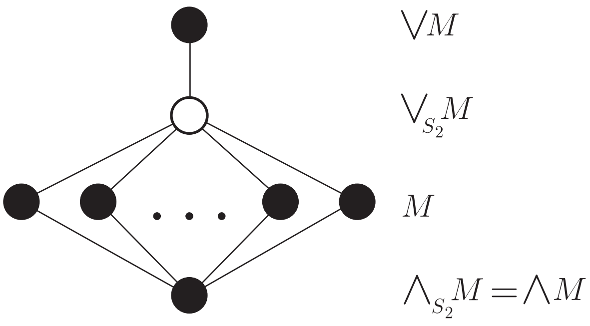


Fig. 1. The structure of $(M_2, \sqsupseteq^*)$ inside $(S_2, \sqsupseteq^*)$. Here M is an arbitrary subset of M_2.

4 Topological properties of M_n

In this section we investigate the topological properties of the set M_n. For this purpose we introduce a metric on this set by following the line of (Hajdu and Hajdu, 2003).

Let n be an integer with $n \geq 2$. The set $\Delta = \{\delta_j \mid \delta_j : \mathbb{Z}^+ \to \mathbb{R},\ j = 1, \ldots, n\}$ is called a weight system if the following three conditions hold:

- $\delta_j(i) > 0 \quad (j \in \{1, \ldots, n\},\ i \in \mathbb{Z}^+)$,
- $\sum\limits_{i=1}^{\infty} \delta_j(i) < \infty \quad (j \in \{1, \ldots, n\})$,
- δ_j is monotone decreasing $(j \in \{1, \ldots, n\})$.

For two sequences $A, B \in S_n$ with $A = (A(i))_{i=1}^{\infty}$ and $B = (B(i))_{i=1}^{\infty}$, put

$$\varrho_\Delta(A, B) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{\infty} |A(i) - B(i)| \delta_j(i).$$

Then (S_n, ϱ_Δ) is a bounded, complete metric space (cf. Theorems 17 and 18 in (Hajdu and Hajdu, 2003)). Moreover, as clearly (S_n, ϱ_Δ) is the product of compact spaces, it is also compact.

We note that the metric ϱ_Δ is defined in this way to fit the behavior of the neighborhood sequences in various subspaces. This is very useful e.g. in connection with the relation $\sqsupseteq^*$. For details see (Hajdu and Hajdu, 2003).

The following statement shows that M_n is an "isolated" subset of S_n.

Theorem 11 *The set $M_n \setminus \{\overline{n}\}$ is a perfect subset of the metric space (S_n, ϱ_Δ).*

PROOF. Let $A \in M_n$ with $A \neq \overline{n}$, and write $A = (A(i))_{i=1}^{\infty}$. Suppose first that A terminates with n-s, that is, for some $k_0 \in \mathbb{Z}^+$ we have $A(k) = n$ whenever $k > k_0$. For each $k > 2k_0$ put $B_k = \overline{A(1) \ldots A(k)}$, and let $B_k = \overline{1}$ for $k = 1, \ldots, 2k_0$. Then by Theorem 1 the B_k are metrics, and clearly $\lim\limits_{k\to\infty} B_k = A$. So A is an accumulation point of $M_n \setminus \{\overline{n}\}$. In the opposite case when A does not terminate with n-s, for every $k \in \mathbb{Z}^+$ put $B_k = A(1) \ldots A(k)\overline{n}$. Then again, the B_k are metrics, and $\lim\limits_{k\to\infty} B_k = A$. This shows that A is an accumulation point of $M_n \setminus \{\overline{n}\}$.

Let now $B \in S_n \setminus M_n$, and write $B = (B(i))_{i=1}^{\infty}$. Then for some $k, l \in \mathbb{Z}^+$ and $j \in \{1, \ldots, n\}$ we have $\sum\limits_{i=1}^{k} B^{(j)}(i) > \sum\limits_{i=l}^{l+k-1} B^{(j)}(i)$. This shows that for any $A \in M_n$ the first $l+k-1$ elements of A cannot be given by $B(1), \ldots, B(l+k-1)$.

Hence $\varrho_\Delta(A,B) \geq \delta_n(l+k-1) > 0$ for all $A \in M_n$, so B is not an accumulation point of $M_n \setminus \{\overline{n}\}$.

Finally, put $B = \overline{n}$. Then, for every $A \in M_n \setminus \{\overline{n}\}$ we have $\varrho_\Delta(A,B) \geq \delta_n(1) > 0$, and the theorem follows. □

Corollary 12 *M_n is a compact subset of (S_n, ϱ_Δ).*

PROOF. The above theorem immediately yields that M_n is closed. As S_n is compact, the statement follows. □

5 Densities of the elements of metrical neighborhood sequences

The densities of the elements can be nicely used to describe the behavior of neighborhood sequences, see e.g. (Hajdu, 2003) for a geometrical characterization. In this section we prove that if $A \in M_n$ then each number from $\{1,\ldots,n\}$ has a density in A. Moreover, we show that these densities can be prescribed arbitrarily. To formulate our results in this direction we need to introduce some further notation.

Let n be an integer with $n \geq 2$. For $A \in S_n$, $k_1, k_2 \in \mathbb{Z}^+$ and $j \in \{1,\ldots,n\}$ let

$$\mathbf{j}(A,k_1,k_2) = \#\{i \mid A(i) = j,\ k_1 \leq i \leq k_2\}.$$

We define the density $s_j(A)$ of the j-s in A as

$$s_j(A) = \lim_{k\to\infty} \frac{\mathbf{j}(A,1,k)}{k},$$

if this limit exists. Finally, for any real number x let $[x]$ denote the integer part of x, i.e. the largest integer which is less than or equal to x.

First we prove that in a metrical neighborhood sequence all elements have densities.

Theorem 13 *For every $A \in M_n$ and $j \in \{1,\ldots,n\}$ the density $s_j(A)$ exists.*

PROOF. Let $A \in M_n$. We proceed by induction. First we prove that $s_1(A)$ exists. Write $s_0 = \liminf_{k\in\mathbb{Z}^+} \frac{\mathbf{1}(A,1,k)}{k}$. We show that $s_1(A) = s_0$. In case of $s_0 = 1$ we are done. Otherwise, let ε be any positive real number. We prove that there exists some integer k_0 such that $k > k_0$ implies $\frac{\mathbf{1}(A,1,k)}{k} < s_0 + \varepsilon$. This is clearly sufficient to prove our statement. By the definition of s_0 we can find an $N \in \mathbb{Z}^+$ such that $\frac{\mathbf{1}(A,1,N)}{N} < s_0 + \frac{\varepsilon}{2}$. Further, let t be an integer with $t > \frac{4}{\varepsilon}$. Put

$k_0 = tN$, and take any integer k with $k > k_0$. Then we can write $k = mN + l$ with $m \geq t$ and $0 \leq l < N$. By the choice of t we have

$$\left|\frac{\mathbf{1}(A,1,k)}{k} - \frac{\mathbf{1}(A,1,mN)}{mN}\right| =$$

$$= \left|\frac{\mathbf{1}(A,mN+1,mN+l)mN - \mathbf{1}(A,1,mN)l}{mN(mN+l)}\right| < \frac{\varepsilon}{2}. \tag{5}$$

On the other hand, for any i with $i = 2, \ldots, m$

$$\mathbf{1}(A,1,N) + \sum_{j=2}^{n} \mathbf{j}(A,1,N) = \mathbf{1}(A,(i-1)N+1,iN) + \sum_{j=2}^{n} \mathbf{j}(A,(i-1)N+1,iN)$$

holds. Moreover, by Theorem 1 we have $\sum\limits_{h=1}^{N} A^{(2)}(h) \leq \sum\limits_{h=(i-1)N+1}^{iN} A^{(2)}(h)$, which implies

$$\mathbf{1}(A,1,N) + 2\sum_{j=2}^{n} \mathbf{j}(A,1,N) \leq \mathbf{1}(A,(i-1)N+1,iN) + 2\sum_{j=2}^{n} \mathbf{j}(A,(i-1)N+1,iN).$$

With the help of the previous two formulas, a simple calculation yields

$$\frac{\mathbf{1}(A,1,N)}{N} \geq \frac{\mathbf{1}(A,1,mN)}{mN}. \tag{6}$$

Combining (5) and (6), we deduce that

$$\frac{\mathbf{1}(A,1,k)}{k} < s_0 + \varepsilon,$$

which proves that $s_1(A)$ exists.

Assume now that for some r with $r \in \{2, \ldots, n\}$ the densities $s_j(A)$ exist for any $j \in \{1, \ldots, r-1\}$. Put $s_0 = \liminf\limits_{k \in \mathbb{Z}^+} \frac{\mathbf{r}(A,1,k)}{k}$. We prove that $s_r(A) = s_0$. In case of $s_0 = 1$ we are done again. Otherwise, take an arbitrary positive ε. Let K be an integer such that for any $k > K$ and $j \in \{1, \ldots, r-1\}$

$$\left|\frac{\mathbf{j}(A,1,k)}{k} - s_j(A)\right| < \frac{\varepsilon}{2(r-1)(r+2)}$$

holds. Fix a positive integer N with $N > K$ such that $\frac{\mathbf{r}(A,1,N)}{N} < s_0 + \frac{\varepsilon}{2}$. Further, take an integer t with $t > \frac{4}{\varepsilon}$, and write $k_0 = tN$. Note that k_0 depends only on ε. Let k be an arbitrary integer with $k > k_0$, and write $k = mN + l$ with $m \geq t$ and $0 \leq l < N$. Then by the choice of t, a simple calculation yields that

$$\left|\frac{\mathbf{r}(A,1,k)}{k} - \frac{\mathbf{r}(A,1,mN)}{mN}\right| =$$

$$= \left| \frac{\mathbf{r}(A, mN+1, mN+l)mN - \mathbf{r}(A, 1, mN)l}{mN(mN+l)} \right| < \frac{\varepsilon}{2}.$$

Clearly, for any $i \in \{1, \dots, m\}$ we have

$$\sum_{j=1}^{n} \mathbf{j}(A, 1, N) = \sum_{j=1}^{n} \mathbf{j}(A, (i-1)N+1, iN).$$

Further, Theorem 1 yields that $\sum_{h=1}^{N} A^{(r+1)}(h) \le \sum_{h=(i-1)N+1}^{iN} A^{(r+1)}(h)$, whence

$$\sum_{j=1}^{r} j \cdot \mathbf{j}(A, 1, N) + (r+1) \sum_{j=r+1}^{n} \mathbf{j}(A, 1, N) \le$$

$$\le \sum_{j=1}^{r} j \cdot \mathbf{j}(A, (i-1)N+1, iN) + (r+1) \sum_{j=r+1}^{n} \mathbf{j}(A, (i-1)N+1, iN).$$

The above assertions by a simple calculation yield that

$$\sum_{j=1}^{r-1} (r+1-j) \left(\frac{\mathbf{j}(A, 1, N)}{N} - \frac{\mathbf{j}(A, 1, mN)}{mN} \right) \ge \frac{\mathbf{r}(A, 1, mN)}{mN} - \frac{\mathbf{r}(A, 1, N)}{N}.$$

By the choice of K and N this immediately gives

$$\frac{\mathbf{r}(A, 1, mN)}{mN} \le \frac{\mathbf{r}(A, 1, N)}{N} + \frac{\varepsilon}{2}.$$

Thus

$$\frac{\mathbf{r}(A, 1, k)}{k} \le s_0 + \varepsilon,$$

which shows that $s_r(A)$ exists. Hence the theorem follows by induction. □

Now we show that the density values can be prescribed arbitrarily, as well.

Theorem 14 *Let n be an integer with $n \ge 2$, and let $\alpha_1, \dots, \alpha_n$ be non-negative real numbers with $\alpha_1 + \dots + \alpha_n = 1$. Then there exists a neighborhood sequence A_n in M_n such that for every $j \in \{1, \dots, n\}$ we have $s_j(A_n) = \alpha_j$.*

PROOF. We prove the theorem by induction on n. For $n = 2$ let A_2 be the unique sequence in S_2 defined by $\mathbf{2}(A_2, 1, k) = [k\alpha_2]$ for each $k \in \mathbb{Z}^+$. By Lemma 2 of (Hajdu and Hajdu, 2004) we know that A_2 is metrical. Moreover, by the definition of A_2 for every $k, l \in \mathbb{Z}^+$ we have that

$$\mathbf{2}(A_2, 1, k) + \mathbf{2}(A_2, 1, l) \le \mathbf{2}(A_2, 1, k+l). \tag{7}$$

Suppose that $t \ge 3$ and $\alpha_1, \dots, \alpha_t$ are given non-negative real numbers with $\alpha_1 + \dots + \alpha_t = 1$. Assume that in the metrical neighborhood sequence $A_{t-1} \in$

S_{t-1} the densities of all the numbers from $\{1,\dots,t-1\}$ exist and we have $s_j(A_{t-1})=\alpha_j$ $(j=1,\dots,t-2)$ and $s_{t-1}(A_{t-1})=\alpha_{t-1}+\alpha_t$. In view of (7) we may further assume that for any $k,l\in\mathbb{Z}^+$

$$(\mathbf{t-1})(A_{t-1},1,k)+(\mathbf{t-1})(A_{t-1},1,l)\le(\mathbf{t-1})(A_{t-1},1,k+l) \qquad (8)$$

holds. If $\alpha_t=0$ then simply put $A_t=A_{t-1}$, and note that A_t is metrical, and $s_j(A_t)=\alpha_j$ $(j=1,\dots,t)$. Otherwise, we define the unique neighborhood sequence $A_t\in S_t$ by replacing some of the $t-1$ elements of A_{t-1} by t such that for every $k\in\mathbb{Z}^+$ we have

$$\mathbf{t}(A_t,1,k)=\left\lfloor(\mathbf{t-1})(A_{t-1},1,k)\frac{\alpha_t}{\alpha_{t-1}+\alpha_t}\right\rfloor.$$

By (8) and the definition of A_t we easily get that for every $k,l\in\mathbb{Z}^+$

$$\mathbf{t}(A_t,1,k)+\mathbf{t}(A_t,1,l)\le\mathbf{t}(A_t,1,k+l)$$

holds, which also shows the validity of (8) for t. Moreover, by Theorem 1 we have

$$\sum_{j=1}^{t-1} j\cdot\mathbf{j}(A_{t-1},1,k)\le\sum_{j=1}^{t-1} j\cdot\left(\mathbf{j}(A_{t-1},l+1,l+k)-\mathbf{j}(A_{t-1},1,l)\right),$$

again for any $k,l\in\mathbb{Z}^+$. Combining the above two inequalities and using the definition of A_t, by a simple calculation we obtain that for every $k,l\in\mathbb{Z}^+$

$$\sum_{j=1}^{t} j\cdot\mathbf{j}(A_t,1,k)\le\sum_{j=1}^{t} j\cdot\left(\mathbf{j}(A_t,l+1,l+k)-\mathbf{j}(A_t,1,l)\right)$$

holds. In view of the metricity of A_i $(i=2,\dots,t-1)$, by Theorem 1 this inequality implies that $A_t\in M_t$. Moreover, a simple calculation yields that we have $s_j(A_t)=\alpha_j$ for each $j\in\{1,\dots,t\}$. Hence the theorem is valid for all $n\ge 2$. □

As a trivial and immediate consequence of the above theorem we obtain that the cardinality of M_n is continuum.

6 Prefixes of metrical neighborhood sequences

In this section we investigate the prefixes of metrical neighborhood sequences. We introduce the following notation. For any positive integer k, let $\mathcal{S}_{n,k}$ denote the set of words of length k, consisting of elements from $\{1,\dots,n\}$. Further, let $\mathcal{M}_{n,k}$ be the subset of $\mathcal{S}_{n,k}$ containing all words which are prefixes of some metrical sequences from M_n. Our first result shows that $\mathcal{M}_{n,k}$ is only a minor subset of $\mathcal{S}_{n,k}$.

Theorem 15 *For any $n \in \mathbb{Z}^+$ we have*

$$\lim_{k\to\infty} \frac{|\mathcal{M}_{n,k}|}{|\mathcal{S}_{n,k}|} = 0.$$

PROOF. First observe that $|\mathcal{M}_{n,k+1}| \leq n \cdot |\mathcal{M}_{n,k}|$ for any $n, k \in \mathbb{Z}^+$. Hence, as $|\mathcal{S}_{n,k}| = n^k$, we have

$$\frac{|\mathcal{M}_{n,k+1}|}{|\mathcal{S}_{n,k+1}|} \leq \frac{|\mathcal{M}_{n,k}|}{|\mathcal{S}_{n,k}|},$$

that is, $|\mathcal{M}_{n,k}|/|\mathcal{S}_{n,k}|$ is monotone decreasing in k.

Let $i, j \in \mathbb{Z}^+$ be arbitrary. Observe that if $A \in \mathcal{M}_{n,ij}$ then either the first j elements of A are 1's, or A does not contain a subword which is a block of j consecutive 1's. This immediately gives

$$|\mathcal{M}_{n,ij}| \leq n^{ij-j} + (n^j - 1)^i.$$

As $|\mathcal{S}_{n,ij}| = n^{ij}$, we have

$$\frac{|\mathcal{M}_{n,ij}|}{|\mathcal{S}_{n,ij}|} \leq \frac{n^{ij-j} + (n^j-1)^i}{n^{ij}} = n^{-j} + \left(1 - \frac{1}{n^j}\right)^i. \tag{9}$$

Now let $\varepsilon > 0$ be arbitrary and choose a $j' \in \mathbb{Z}^+$ such that $n^{-j'} < \varepsilon/2$, and then an $i' \in \mathbb{Z}^+$ with $(1 - 1/n^{j'})^{i'} < \varepsilon/2$. Then (9) implies that

$$\frac{|\mathcal{M}_{n,i'j'}|}{|\mathcal{S}_{n,i'j'}|} < \varepsilon.$$

By the monotonity of $|\mathcal{M}_{n,k}|/|\mathcal{S}_{n,k}|$ the proof is complete. $\square$

Remark 16 *For any $n, k \in \mathbb{Z}^+$ let $M_{n,k}$ denote the set of metrical neighborhood sequences from M_n, having a period of length k. As clearly $|M_{n,k}| \leq |\mathcal{M}_{n,k}|$ we also have*

$$\lim_{k\to\infty} \frac{|M_{n,k}|}{|\mathcal{S}_{n,k}|} = 0$$

for any $n \in \mathbb{Z}^+$.

Now we present some numerical data which strongly indicate that in spite of the above theorem, the number of elements of $M_{n,k}$ and $\mathcal{M}_{n,k}$ grow exponentially with k. (Note that $|\mathcal{S}_{n,k}| = n^k$.) We mention that in 2D, Das et al.

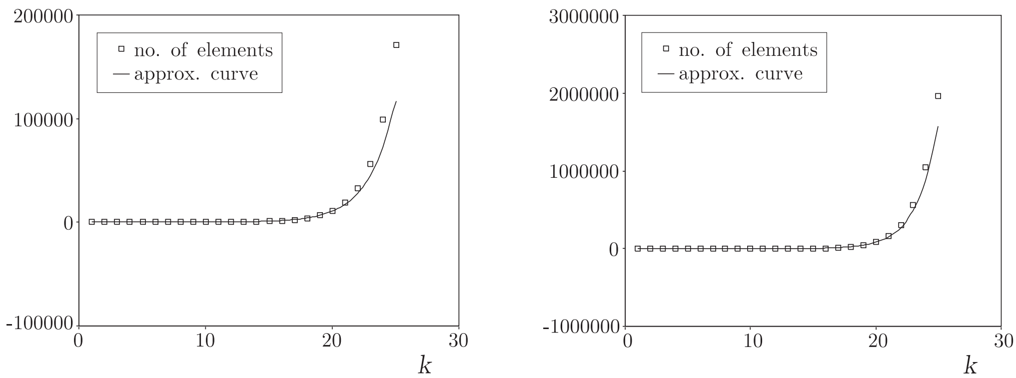
(1987a) have presented similar data for smaller range. For 2D sequences the results of our calculations are summarized in Table 2.

Table 2
Number of elements of $M_{2,k}$ and $\mathcal{M}_{2,k}$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\lvert\mathcal{S}_{2,k}\rvert$	2	4	8	16	32	64	128	256	512	1 024	2 048	4 096	8 192	16 384	32 768	65 536
$\lvert M_{2,k}\rvert$	2	3	4	6	8	13	18	29	44	71	110	181	290	483	790	1 330
$\lvert\mathcal{M}_{2,k}\rvert$	2	3	5	8	14	23	41	70	125	218	395	697	1 273	2 279	4 185	7 568

k	17	18	19	20	21	22	23	24	25
$\lvert\mathcal{S}_{2,k}\rvert$	131 072	262 144	524 288	1 048 576	2 097 152	4 194 304	8 388 608	16 777 216	33 554 432
$\lvert M_{2,k}\rvert$	2 212	3 776	6 360	10 982	18 704	32 611	56 080	98 598	171 068
$\lvert\mathcal{M}_{2,k}\rvert$	13 997	25 500	47 414	87 024	162 456	299 947	562 345	1 043 212	1 962 589

Table 2 suggests that the number of elements in $M_{2,k}$ and $\mathcal{M}_{2,k}$ grow exponentially. Based upon our data, using the software package SPSS® [1] we obtained the approximations shown in Figure 2. We find that the exponential functions $0.7501 \cdot \exp(0.4783 \cdot k)$, and $0.7541 \cdot \exp(0.5821 \cdot k)$ fit well to the cardinality of the sets $M_{2,k}$, and $\mathcal{M}_{2,k}$, respectively.



(a) (b)

Fig. 2. The exponential increment of the number of (a) $M_{2,k}$ approximated by $0.7501 \cdot \exp(0.4783 \cdot k)$, (b) $\mathcal{M}_{2,k}$ approximated by $0.7541 \cdot \exp(0.5821 \cdot k)$.

For interest we mention that $\exp(0.4783) = 1.6133\ldots$ is rather close to the golden ratio $(1 + \sqrt{5})/2 = 1.6180\ldots$, and that there might be some connection between the sequence $M_{2,k}$ and the Fibonacci sequence. This relation is somewhat supported also by Table 2.

Finally, in the following Table 3 we give some data concerning the higher dimensional cases. Based on this table, it is very probable that both $M_{n,k}$ and $\mathcal{M}_{n,k}$ grow exponentially in k, for any fixed $n \in \mathbb{Z}^+$.

[1] SPSS for Windows 6.0+ Base System, Regression Models, SPSS Inc., Chicago.

Table 3
Number of elements of $M_{n,k}$ and $\mathcal{M}_{n,k}$ for $3 \leq n \leq 8$.

k	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\mathcal{S}_{3,k}\rvert$	3	9	27	81	243	729	2 187	6 561	19 683	59 049	177 147	531 441
$\lvert M_{3,k}\rvert$	3	6	10	20	34	74	136	295	606	1 329	2 839	6 480
$\lvert\mathcal{M}_{3,k}\rvert$	3	6	14	31	77	179	456	1 115	2 879	7 258	19 115	49 090
$\lvert\mathcal{S}_{4,k}\rvert$	4	16	64	256	1 024	4 096	16 384	65 536	262 144	1 048 576		
$\lvert M_{4,k}\rvert$	4	10	20	50	103	280	636	1 737	4 439	12 319		
$\lvert\mathcal{M}_{4,k}\rvert$	4	10	30	85	273	820	2 711	8 612	29 015	95 482		
$\lvert\mathcal{S}_{5,k}\rvert$	5	25	125	625	3 125	15 625	78 125	390 625				
$\lvert M_{5,k}\rvert$	5	15	35	105	254	826	2 230	7 328				
$\lvert\mathcal{M}_{5,k}\rvert$	5	15	55	190	748	2 754	11 181	43 652				
$\lvert\mathcal{S}_{6,k}\rvert$	6	36	216	1 296	7 776	46 656	279 936					
$\lvert M_{6,k}\rvert$	6	21	56	196	544	2 058	6 425					
$\lvert\mathcal{M}_{6,k}\rvert$	6	21	91	371	1 729	7 536	36 259					
$\lvert\mathcal{S}_{7,k}\rvert$	7	49	343	2 401	16 807	117 649						
$\lvert M_{7,k}\rvert$	7	28	84	336	1 052	4 536						
$\lvert\mathcal{M}_{7,k}\rvert$	7	28	140	658	3 542	17 833						
$\lvert\mathcal{S}_{8,k}\rvert$	8	64	512	4 096	32 768	262 144						
$\lvert M_{8,k}\rvert$	8	36	120	540	1 882	9 108						
$\lvert\mathcal{M}_{8,k}\rvert$	8	36	204	1 086	6 630	37 859						

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